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Path Convexities on Graphs

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1 Introduction

A feasible family of paths in a connected graph G is a family that contains at least one path between any pair of vertices in G. Any feasible path family defines a transit function on G. Well-known instances are: the geodesics, the induced paths, and all paths. Such transit functions are called *path transit functions* on G in [12]. A prime example is the interval function $I : V \times V \to 2^V$ of a connected graph G, where $I(u, v)$ is the set of the vertices lying on shortest paths between u and v. Other examples are the induced path transit function, and the all-paths transit function. Any transit function on (V, σ) defines a natural convexity on V. The convexities associated with the three mentioned path transit functions have already been studied extensively. Some relevant references are: for the geodesic convexity [5],[11],[14], [17], [10], for the induced path convexity [4], [13], [8], [15], for triangle path convexity, [2] and for the all-paths convexity (or the coarse convexity) [16], [3], [1]. See also the recent survey [7].

We consider mainly the above mentioned three examples and transit functions constructed from these. By choosing the perspective of transit functions we propose a unifying approach for the study of such path properties. This approach suggests also various new questions for future research. We discuss the behaviour of the associated convexities and the classical convexity invariants, such as the Carathéodory, Helly and Radon numbers in relation with graph invariants, such as the clique number and other graph properties.

Our graphs $G = (V, E)$ here are connected, simple and loopless. A *transit function* on G is a function $R : V \times V \to 2^V$ satisfying the following three axioms:

(*t*1) $u \in R(u, v)$ for all u and v in V,

(t2) $R(u, v) = R(v, u)$ for all u and v in V.
(t3) $R(u, u) = \{u\}$.

A transit function R on V is a betweenness, if it satisfies the betweenness axioms

(b1) $x \in R(u, v),\ x \neq v \Longrightarrow v \notin R(u, x)$,
(b2) $x \in R(u, v) \Longrightarrow R(u, x) \subseteq R(u, v)$.

Remark: Any function $R : V \times V \to 2^V$ satisfying $(t1)$, $(t2)$ and $(b1)$ is a transit function.

A subset W of V is *R-convex* if $R(u, v) \subseteq W$, for all u, v in W. A subgraph is convex if it is induced by a convex set. The family $\mathcal{C}_R$ of all R-convex sets in V is an abstract convexity: it is closed under intersections and nested unions, and both $\emptyset$ and V are R-convex. Note that, in the finite case, the condition on nested unions can be deleted. The convexity $\mathcal{C}_0$ of the discrete transit function 0 is the *discrete convexity*: every subset is convex. The convexity $\mathcal{C}_1$ of the trivial transit function 1 is the *trivial convexity*. Note that we assume that singletons are always convex. This is no real restriction of the notion of a convexity, because if we add all missing singletons to a convexity, then it remains a convexity. Thus the empty set $\emptyset$, the singletons $\{u\}$ and the whole set V are the *trivial convex sets* of a convexity. The smallest R-convex subset containing a subset W of V is denoted by $\langle W \rangle_R$ and is called the *R-convex hull* of W. Note that two different transit functions R and S may give rise to the same convexity, that is, $\mathcal{C}_R = \mathcal{C}_S$. Convexities defined by a transit functions are called interval convexities, or interval spaces in e.g. [6] and [17].

The *Carathéodory number c* of the convexity space $\mathcal{C}$ is the smallest integer (if it exists) such that for any finite subset F of V, $\langle F \rangle_{\mathcal{C}} = \bigcup\{\langle S \rangle_{\mathcal{C}} \mid S \subseteq F, |S| \leq c\}$. The *exchange number e* of $\mathcal{C}$ is the smallest integer (if it exists) such that for any subset F of V with $|F| \geq e$ and any point p in F, we have $\langle F - p \rangle_{\mathcal{C}} \subseteq \bigcup\{\langle F - a \rangle_{\mathcal{C}} \mid a \in F - p\}$. The *Helly number h* of $\mathcal{C}$ is the smallest integer (if it exists) such that every family of convex sets with an empty intersection contains a subfamily of at most h members with an empty intersection. Equivalently, h is the smallest natural number such that $\bigcap_{s \in F} \langle F - s \rangle_{\mathcal{C}} \neq \emptyset$ for every $(h+1)$-element subset F of V. The *Radon number r* of $\mathcal{C}$ is the smallest integer (if it exists) such that every r-element set $S \subseteq V$ admits a Radon partition, that is, a partition $S = S_1 \cup S_2$, $(S_1 \cap S_2 = \emptyset)$ with $\langle S_1 \rangle_{\mathcal{C}} \cap \langle S_2 \rangle_{\mathcal{C}} \neq \emptyset$. The m^{th} *Radon number*, denoted by r_m, is the smallest number (if it exists) such that every r_m-element set $W \subseteq V$ admits a Radon m-partition, that is a partition of S into m pairwise disjoint subsets $W_1, W_2, ..., W_m$ such that $\langle W_1 \rangle_{\mathcal{C}} \cap \langle W_2 \rangle_{\mathcal{C}} \cap .... \cap \langle W_m \rangle_{\mathcal{C}} \neq \emptyset$. A subset $S \subseteq V$ is called a *convex-independent set* if $x \notin \langle S - x \rangle_{\mathcal{C}}$ for every $x \in S$. The *rank* of $\mathcal{C}$ is the supremum of the cardinalities of the independent subsets of V. The *hull number u* of $\mathcal{C}$ is the infimum of the cardinalities of subsets S of V such that $\langle S \rangle_{\mathcal{C}} = V$. There are very many literature available on convexity invariants of graph convexities; to mention some of the most important ones [9], [4, 5] Let Φ be a property of paths in G, for instance the property of being a *geodesic*, i.e. a shortest path. A Φ-*path* is a path having property Φ. Formally, we take a *path property* Φ to be a subset of the set of all paths in G. Thus, if P is a Φ-path, then we may denote that also as $P \in \Phi$. Let u and v be vertices of G. Then $\Phi(u, v)$ denotes the subset of all u, v-paths in Φ. Here We consider *feasible* path properties, that is, path properties Φ such that $\Phi(u, v) \neq \emptyset$, for all u, v in V. The Φ-*path transit function*, or Φ-*path function* for short, on G is the transit function R_Φ defined by

$$R_\Phi(u, v) = \{x \in V \mid x \text{ is on some } \Phi\text{-path in } G\ \}.$$

Note that the subgraph induced by $R_\Phi(u, v)$ is a connected subgraph of G. If no confusion arises, we call a Φ-path transit function a path transit function. The convexity $\mathcal{C}_{R_\Phi}$ will also be denoted as $\mathcal{C}_\Phi$.

If R_{Φ_1} and R_{Φ_2} are two path transit functions, then $R_{\Phi_1} \wedge R_{\Phi_2}$ need not be a path transit function. For example, if Φ_1 = 'shortest' and Φ_2 = 'longest', then $R_{\Phi_1} \wedge R_{\Phi_2}$ usually will not be a path transit function. However, $R_{\Phi_1} \vee R_{\Phi_2}$ is always a path transit function, namely of the path property $\Phi = \Phi_1 \cup \Phi_2$. Hence, the family of the path transit functions on G is a join semi-lattice of L_G, denoted by $L_{p(G)}$. Clearly, the *all-paths transit function* on G defined by

$$A_G(u, v) = \{x \in V \mid x \text{ lies in some } u, v\text{-path in } G\},$$

is a universal upper bound of $L_{p(G)}$.

2 Examples of path properties

We survey the well studied path transit functions and their associated convexities together with some new path transit functions that may be of interest to Graph Theorists. We focus our attention to the betweennes and monotone properties of these transit functions. For the associated convexities, we discuss the convex hull and other convexity properties like JHC and convexity and graph theoretical invariants.

We mainly discuss the following Path transit functions

- **The geodesic transit function**
- **The induced path transit function**
- **The all-paths transit function**
- **The I_j-path transit function**
- **The triangle-path transit functions**
- **The Longest Path (Detour)transit function**

References

[1] M.Changat, S. Klavžar, and H.M. Mulder, The all-paths transit function of a graph, Czech.Math J. 51 (126), (2001), 439–448.

[2] M.Changat, J.Mathew, On triangle path convexity in graphs, Discrete Math. 206 (1999), 91–95.

[3] P. Duchet, Convexity in combinatorial structures, Rend. Circ. Mat. Palermo (2) Suppl. 14 (1987), 261–293.

[4] P. Duchet, Convex sets in graphs II. Minimal path convexity, J. Combin. Theory Ser. B. 44 (1988), 307–316.

[5] P. Duchet, Discrete convexity: retractions, morphisms and the partition problem, in: Proc. of the conf. on graph connections, India, 1998, Allied Publishers, New Delhi.

[6] J. Calder, Some elementary properties of interval convexities, J. London Math. Soc. 3 (1971), 422–428.

[7] M.Changat, H.M.Mulder and G.Sierksma, Convexities related to path properties on graphs, Discrete Math. 290 (2005), 117-131.

[8] M.Changat, J.Mathew, Induced path transit function, monotone and Peano axioms, Discrete Math. 286 (2004), 185-194.

[9] R.E. Jamison, Partition numbers for trees and ordered sets, Pacific J. Math. 96 (1981), 115-140.

[10] S. Klavžar, H.M. Mulder, Median graphs; characterizations, location theory and related structures, J. Combin. Math. Combin. Comp. 30 (1999), 103–127.

[11] H.M. Mulder, The interval function of a graph, Mathematical Centre Tracts 132, Mathematisch Centrum, Amsterdam, 1980.

[12] H.M. Mulder, Transit functions on graphs, in preparation.

[13] M.A. Morgana, H.M. Mulder, The induced path convexity, betweenness and svelte graphs, to appear in Discrete Math.

[14] L. Nebeský, A characterization of the interval function of a (finite or infinite) connected graph, Czech. Math. J. 51 (126) (2001), 635–642.

[15] L. Nebeský, The induced paths in a connected graph and a ternary relation determined by them, Mathematica Bohemica 127 (2002), 397–408.

[16] E. Sampathkumar, Convex sets in graphs, Indian J. Pure Appl. Math. 15 (1984), 1065–1071.

[17] M.L.J. van de Vel, Theory of convex structures, North Holland, Amsterdam, 1993.