On the Metric Dimension of Cartesian Products of Graphs

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This is an extended abstract of a paper on resolving sets[1] in cartesian products[2] of graphs. The full paper is at `http://www.arxiv.org/math/0507527`.

§1. Hypercubes. The *hypercube* Q_n is the graph whose vertices are the n-dimensional binary vectors, where two vertices are adjacent if they differ in exactly one coordinate. It is well known that Q_n is the n-fold cartesian product $K_2 \,\square\, K_2 \,\square\, \cdots \,\square\, K_2$. It is easily seen that $\beta(Q_n) \leq n$. The first case when this bound is not tight is $n = 5$. A laborious calculation verifies that Q_5 is resolved by the 4-vertex set $\{00000, 00011, 00101, 01001\}$. We have determined $\beta(Q_n)$ for small values of n by computer search.

n	2	3	4	5	6	7	8	10	15
$\beta(Q_n)$	2	3	4	4	5	6	6	≤ 7	≤ 10

The asymptotic value of $\beta(Q_n)$ turns out to be related to the following coin weighing problem first posed by Söderberg and Shapiro in 1963. Given n coins, each with one of two distinct weights, determine the weight of each coin with the minimum number of weighings. We are interested in the static variant of this problem, where the choice of sets of coins to be weighed is determined in advance. Weighing a set S of coins determines how many light (and heavy) coins are in S, and no further information. It follows that the minimum number of weighings differs from $\beta(Q_n)$ by at most 1. A lower bound on the number of weighings by Erdős and Rényi (1963) and an upper bound by Lindström (1964) imply that

$$\lim_{n\to\infty} \beta(Q_n) \cdot \frac{\log n}{n} = 2,$$

where, as always in this paper, logarithms are binary. Note that Lindström's proof is constructive. He gives an explicit scheme of $2^k - 1$ weighings that suffice for $k \cdot 2^{k-1}$ coins.

§2. Mastermind and Hamming Graphs: *Mastermind* is a game for two players, the *code setter* and the *code breaker*. The code setter chooses a secret vector $s = [s_1, s_2, \ldots, s_n] \in \{1, 2, \ldots, k\}^n$. The task of the code breaker is to infer the secret vector by a series of questions, each a vector $t = [t_1, t_2, \ldots, t_n] \in \{1, 2, \ldots, k\}^n$. The code setter answers with two integers, first being the number of positions in which the secret vector and the question agree, denoted by $a(s,t) = |\{i : s_i = t_i, 1 \leq i \leq n\}|$. The second integer $b(s,t)$ is the maximum of $a(\tilde{s}, t)$, where $\tilde{s}$ ranges over all permutations of s.

In the commercial version of the game, $n = 4$ and $k = 6$. The secret vector and each question is represented by four pegs each coloured with one of six colours. Each answer is represented by

[1] Let G be a graph with vertex $V(G)$ and edge set $E(G)$. The distance between vertices $v, w \in V(G)$ is denoted by $d_G(v,w)$, or $d(v,w)$ if the graph G is clear from the context. A vertex $x \in V(G)$ *resolves* a pair of vertices $v, w \in V(G)$ if $d(v,x) \neq d(w,x)$. A set of vertices $S \subseteq V(G)$ *resolves* G, and S is a *resolving set* of G, if every pair of distinct vertices of G are resolved by some vertex in S. A resolving set S of G with the minimum cardinality is a *metric basis* of G, and $|S|$ is the *metric dimension* of G, denoted by $\beta(G)$.

[2] The *cartesian product* of graphs G and H, denoted by $G \,\square\, H$, is the graph with vertex set $V(G) \times V(H) := \{(a,v) : a \in V(G), v \in V(H)\}$, where (a,v) is adjacent to (b,w) whenever $a = b$ and $\{v,w\} \in E(H)$, or $v = w$ and $\{a,b\} \in E(G)$. Where there is no confusion the vertex (a,v) of $G \,\square\, H$ will be written av. Observe that if G and H are connected, then $G \,\square\, H$ is connected. In particular, $d(av, bw) = d_G(a,b) + d_H(v,w)$ for all vertices av, bw of $G \,\square\, H$.

$a(s,t)$ black pegs, and $b(s,t) - a(s,t)$ white pegs. Knuth (1976) showed that four questions suffice to determine s in this case. Here the code breaker may determine each question in response to the previous answers. *Static mastermind* is the variation in which all the questions must be supplied at once. Let $g(n,k)$ denote the maximum, taken over all vectors s, of the minimum number of questions required to determine s in this static setting.

The *Hamming graph* $H_{n,k}$ is the n-fold cartesian product of cliques $K_k \,\square\, K_k \,\square \cdots \square\, K_k$. Note that the hypercube $Q_n = H_{n,2}$. The vertices of $H_{n,k}$ can be thought of as vectors in $\{1,2,\dots,k\}^n$, with two vertices being adjacent if they differ in precisely one coordinate. Thus the distance $d_H(v,w)$ between two vertices v and w is the number of coordinates in which their vectors differ. That is, $d_H(v,w) = n - a(v,w)$.

Suppose for the time being that we remove the second integer $b(s,t)$ from the answers given by the code setter in the static mastermind game. Let $f(n,k)$ denote the maximum, taken over all vectors s, of the minimum number of questions required to determine s without $b(s,t)$ in the answers. For the code breaker to correctly infer the secret vector s from a set of questions T, s must be uniquely determined by the values $\{a(s,t) : t \in T\}$. Equivalently, for any two vertices v and w of $H_{n,k}$, there is a $t \in T$ for which $a(v,t) \neq a(w,t)$; that is, the distances $d_H(v,t) \neq d_H(w,t)$. Hence the secret vector can be inferred if and only if T resolves $H_{n,k}$. Thus $g(n,k) \le f(n,k) = \beta(H_{n,k})$. Chvátal (1983) proved the upper bound $\beta(H_{n,k}) = f(n,k) \le (2+\epsilon) n \frac{1+2\log k}{\log n - \log k}$ for large $n > n(\epsilon)$ and small $k < n^{1-\epsilon}$.

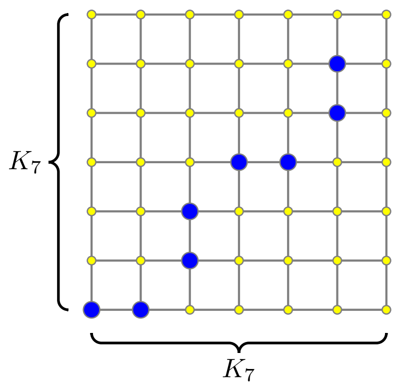

FIGURE 1. Resolving set of $K_7 \,\square\, K_7$.

We study $\beta(H_{n,k})$ for large values of k rather that for large values of n. We prove $\beta(H_{2,k}) = \left\lfloor \frac{2}{3}(2k-1) \right\rfloor$ for all $k \ge 1$. This result is a special case (with $m = n = k$) of the following more general result: For all $n \ge m \ge 1$,

$$\beta(K_n \,\square\, K_m) = \begin{cases} \left\lfloor \frac{2}{3}(n+m-1) \right\rfloor & \text{, if } m \le n \le 2m-1 \\ n-1 & \text{, if } n \ge 2m-1. \end{cases}$$

§3. DOUBLY RESOLVING SETS: Many of the results that follow are based on the following idea. Let $G \neq K_1$ be a graph. Two vertices $v, w \in V(G)$ are *doubly resolved* by $x, y \in V(G)$ if

$$d(v,x) - d(w,x) \neq d(v,y) - d(w,y).$$

A set of vertices $S \subseteq V(G)$ *doubly resolves* G, and S is a *doubly resolving set*, if every pair of distinct vertices $v, w \in V(G)$ are doubly resolved by two vertices in S. Every graph with at least two vertices has a doubly resolving set. Let $\psi(G)$ denote the minimum cardinality of a doubly resolving set of a graph $G \neq K_1$. Note that if x, y doubly resolves v, w then $d(v,x) - d(w,x) \neq 0$ or $d(v,y) - d(w,y) \neq 0$, and at least one of x and y resolves v, w. Thus a doubly resolving set is also a resolving set, and $\beta(G) \le \psi(G)$. Doubly resolving sets are interesting because of the following upper bound for G and $H \neq K_1$:

$$\beta(G \,\square\, H) \le \beta(G) + \psi(H) - 1. \qquad (1)$$

Proof. Let S be a metric basis of G. Let T be a doubly resolving set of H with $|T| = \psi(H)$. Fix vertices $s \in S$ and $t \in T$. Let $X := \{sv : v \in T\} \cup \{at : a \in S\}$. Observe that $|X| = |S| + |T| - 1$. Consider two vertices av and bw of $G \,\square\, H$. If $a = b$ then av and bw are resolved since the projection[3] of X onto H is T. Similarly, if $v = w$ then av and bw are resolved since the projection of X onto G is S. Now assume that $a \neq b$ and $v \neq w$. Since T is doubly resolving for H, there are two vertices $x, y \in T$ such that $d_H(v,x) - d_H(w,x) \neq d_H(v,y) - d_H(w,y)$. Thus for at least one of x and y, say x, we have $d_H(v,x) - d_H(w,x) \neq d_G(b,s) - d_G(a,s)$. Hence $d(av, sx) = d_G(a,s) + d_H(v,x) \neq d_G(b,s) + d_H(w,x) = d(bw, sx)$. That is, $sx \in X$ resolves av and bw. □

Moreover, we have a lower bound. Suppose that S resolves $G \,\square\, G$ for some graph G. Let A and B be the two projections of S onto G. Then $A \cup B$ doubly resolves G. In particular,

$$\beta(G \,\square\, G) \geq \tfrac{1}{2}\psi(G). \tag{2}$$

Equations (1) and (2) prove that $\beta(G \,\square\, G)$ is always within a constant factor of $\psi(G)$. In particular,

$$\tfrac{1}{2}\psi(G) \leq \beta(G \,\square\, G) \leq \psi(G) + \beta(G) - 1 \leq 2\psi(G) - 1.$$

Thus doubly resolving sets are essential in the study of the metric dimension of cartesian products. A natural candidate for a resolving set of $G \,\square\, G$ is $S \times S$ for a well chosen set $S \subseteq V(G)$. It follows from (2) and the proof of (1) that $S \times S$ resolves $G \,\square\, G$ if and only if S doubly resolves G.

§4. Complete Graphs: It is well known that if G has n vertices then $\beta(G) = n - 1$ if and only if $G = K_n$. We prove that $\psi(K_n) = \max\{n-1, 2\}$ for all $n \geq 2$. Thus (1) implies that $\beta(K_n \,\square\, G) \leq \beta(G) + \max\{n-2, 1\}$ for every graph G. In certain cases, this result can be improved to $\beta(K_n \,\square\, G) \leq \max\{n-1, 2 \cdot \beta(G)\}$. Thus if n is large, in particular when $n \geq 2 \cdot \beta(G) + 1$, then

$$\beta(K_n \,\square\, G) = n - 1. \tag{3}$$

§5. Paths and Grids: Khuller et al. (1996) and Chartrand et al. (2000) proved that

$$\beta(G) = 1 \text{ if and only if } G = P_n \text{ for some } n. \tag{4}$$

Thus $\beta(K_n \,\square\, P_m) = n - 1$ by (3) for all $n \geq 3$. It is easily seen that $\psi(P_n) = 2$ for all $n \geq 2$. Thus (1) implies that every graph G satisfies

$$\beta(G) \leq \beta(G \,\square\, P_n) \leq \beta(G) + 1, \tag{5}$$

as proved by Chartrand et al. (2000) in the case that $n = 2$.

An n-dimensional *grid* is a cartesian product of paths $P_{m_1} \,\square\, P_{m_2} \,\square\, \cdots \,\square\, P_{m_n}$. Equations (4) and (5) imply that $\beta(P_{m_1} \,\square\, P_{m_2} \,\square\, \cdots \,\square\, P_{m_n}) \leq n$, as proved by Khuller et al. (1996), who in addition claimed that $\beta(P_{m_1} \,\square\, P_{m_2} \,\square\, \cdots \,\square\, P_{m_n}) = n$. They wrote 'we leave it for the reader to see why n is a lower bound'. This claim is false if every $m_i = 2$ and n is large, since $\beta(P_2 \,\square\, P_2 \,\square\, \cdots \,\square\, P_2) \to 2n/\log n$.

§6. Cycles: Khuller et al. (1996) and Saenpholphat and Zhang(2004) observed that $\beta(C_n) = 2$ for $n \geq 3$. We prove that for all $n \geq 3$,

$$\psi(C_n) = \begin{cases} 2 & \text{, if } n \text{ is odd} \\ 3 & \text{, if } n \text{ is even.} \end{cases} \tag{6}$$

Equations (1) and (6) imply that every graph G satisfies

$$\beta(G) \leq \beta(G \,\square\, C_n) \leq \begin{cases} \beta(G) + 1 & \text{, if } n \text{ is odd} \\ \beta(G) + 2 & \text{, if } n \text{ is even.} \end{cases}$$

[3] Let $S \subseteq V(G \,\square\, H)$ for graphs G and H. Then every pair of vertices in a fixed row of $G \,\square\, H$ are resolved by S if and only if the projection of S onto G resolves G (and similarly for columns). It follows that for all graphs G and H, and for every resolving set S of $G \,\square\, H$, the projection of S onto G resolves G, and the projection of S onto H resolves H. In particular, $\beta(G \,\square\, H) \geq \max\{\beta(G), \beta(H)\}$.

Now consider the cartesian product of a path and a cycle. We have $\beta(P_m \square C_n) \leq \psi(C_n) + \beta(P_m) - 1 \leq 3 + 1 - 1 = 3$ by (6) and (4). We prove that for $n \geq 3$, $\beta(G \square C_n) = 2$ if and only if G is a path and n is odd. Thus for $m \geq 2$ and $n \geq 3$,

$$\beta(P_m \square C_n) = \begin{cases} 2 & \text{, if } n \text{ is odd} \\ 3 & \text{, if } n \text{ is even.} \end{cases}$$

For the product of two cycles ($m, n \geq 3$) we prove

$$\beta(C_m \square C_n) = \begin{cases} 3 & \text{, if } m \text{ or } n \text{ is odd} \\ 4 & \text{, otherwise.} \end{cases}$$

For the product of a cycle ($m \geq 3$) and a complete graph ($n \geq 1$) we prove

$$\beta(K_n \square C_m) = \begin{cases} 2 & \text{, if } n = 1 \text{, or } n = 2 \text{ and } m \text{ is odd} \\ 3 & \text{, if } n = 2 \text{ and } m \text{ is even, or } n = 3 \text{, or } n = 4 \text{ and } m \text{ is even} \\ 4 & \text{, if } n = 4 \text{ and } m \text{is odd} \\ n - 1 & \text{, if } n \geq 5. \end{cases}$$

§7. Trees: Let v be a vertex of a tree T. Let ℓ_v be the number of components of $T \setminus v$ that are (possibly edgeless) paths. Khuller et al. (1996) and Chartrand et al. (2000) proved that for every tree T that is not a path,

$$\beta(T) = \sum_{v \in V(T)} \max\{\ell_v - 1, 0\}. \tag{7}$$

Doubly resolving sets in trees are well-understood. In fact, the unique minimum doubly resolving set in a tree T is the set of leaves of T. Thus, (1) implies that if a tree T has k leaves then $\beta(T \square G) \leq \beta(G) + k - 1$ for every graph G. Moreover, many leaves in a graph force up the metric dimension of its cartesian product. In particular, $\beta(G \square G) \geq k$ for every graph G with $k \geq 2$ leaves. For $n \geq 4$, let B_n be the *comb* graph obtained by attaching one leaf at every vertex of P_n.

$$\beta(B_n) = 2 \text{ and } n = \psi(B_n) \leq \beta(B_n \square B_n) \leq n + 1. \tag{8}$$

Proof. Now $\ell_v = 0$ for every leaf v of B_n, and $\ell_w = 1$ for every other vertex w of B_n, except for the two vertices x and y indicated in Figure 2, for which $\ell_x = \ell_y = 2$. Thus $\beta(B_n) = 2$ by (7). Since B_n has n leaves, by the above discussion, $\psi(B_n) = n$ and $\beta(B_n \square B_n) \geq n$. The upper bound $\beta(B_n \square B_n) \leq n + 1$ follows from (1). □

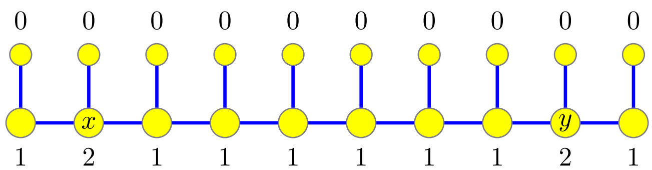

Figure 2. An illustration of the comb graph B_{10} showing the ℓ-values at each vertex.

Equation (8) implies that ψ is not bounded by any function of metric dimension; this is one of the major contributions of the paper. Equation (8) generalises as follows: For all $k \geq 1$ and $n \geq 2$ there is a k-connected graph $G_{n,k}$ for which $\beta(G_{n,k}) \leq 2k$ and $\beta(G_{n,k} \square G_{n,k}) \geq n$.

Open problem: Prove analogous results for direct and strong products of graphs.