Irreducibility of the Tutte Polynomial of a Connected Matroid

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We solve in the affirmative a conjecture of Brylawski, namely that the Tutte polynomial of a connected matroid is irreducible over the integers. © 2001 Elsevier Science

If \( M \) is a matroid over a set \( E \), then its Tutte polynomial is defined as

\[
T(M; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A| - r(A)},
\]

where \( r(A) \) is the rank of \( A \) in \( M \). This polynomial is an important invariant as it contains much information on the matroid; see [2, 3] for useful surveys.

One of the basic properties of \( T(M; x, y) \) is that, if \( M \) is the direct sum of two matroids \( M_1 \) and \( M_2 \), then

\[
T(M; x, y) = T(M_1; x, y) T(M_2; x, y).
\]

In particular, this implies that \( T(M; x, y) \) has a non-trivial factor in \( \mathbb{Z}[x, y] \) if \( M \) is disconnected. Brylawski [1] conjectured that the converse also holds; this paper is devoted to a proof of this conjecture.

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**Theorem 1.** If $M$ is a connected matroid, then $T(M; x, y)$ is irreducible in $\mathbb{Z}[x, y]$.

Actually, an analysis of our proof shows that $T(M; x, y)$ is irreducible even in $\mathbb{C}[x, y]$.

**Corollary 1.** If a matroid $M$ has $c$ connected components $M_1, \ldots, M_c$, then the factorization of $T(M; x, y)$ in $\mathbb{Z}[x, y]$ is exactly

$$T(M; x, y) = T(M_1; x, y) \cdots T(M_c; x, y).$$

The main tool in proving our result is the following set of linear equations $(B_k)$ that are satisfied by the coefficients of the Tutte polynomial of any matroid, and that were proved in [1].

**Lemma 1.** Let $T(M; x, y) = \sum b_{ij} x^i y^j$ be the Tutte polynomial of a matroid $M$ and let $m$ be the number of elements in $M$. Then

$$\sum_{i=0}^k \sum_{s=0}^{k-s} (-1)^s \binom{k-s}{t} b_{st} = 0, \quad (B_k)$$

for $k = 0, 1, \ldots, m-1$.

We also need the following basic properties [2]:

1. $b_{00} = 0$ if $|E| \geq 1$;
2. $b_{10} \neq 0$ if and only if $M$ is connected;
3. $x^k \mid T(M; x, y)$ if and only if $M$ has at least $k$ coloops;
4. $y^k \mid T(M; x, y)$ if and only if $M$ has at least $k$ loops;
5. If $i \geq r(M)$ or $j \geq n(M)$, then $b_{ij} = 0$, except if $i = r(M)$ and $j = 0$, or if $i = 0$ and $j = n(M)$, where $r(M)$ is the rank of $M$ and $n(M) = m - r(M)$ is the nullity of $M$.

Suppose now $M$ is a connected matroid on a set of $m$ elements, and that there is a non-trivial factorization

$$T(M; x, y) = \sum b_{ij} x^i y^j = A(x, y) C(x, y), \quad (1)$$

where $A(x, y) = \sum a_{ij} x^i y^j$ and $C(x, y) = \sum c_{ij} x^i y^j$.

Since $b_{00} = 0$, either $a_{00}$ or $c_{00}$ is zero; we may assume $a_{00} = 0$. Since

$$0 \neq b_{10} = a_{00} c_{10} + a_{10} c_{00},$$
the assumption implies that \( c_{00} \neq 0 \). We will prove that \( c_{00} = 0 \), thus obtaining a contradiction.

Since \( M \) is connected, by properties (3) and (4) above, neither \( x \) nor \( y \) are factors of \( A(x, y) \) or \( C(x, y) \). Define for a polynomial \( P(x, y) = \sum p_{ij}x^iy^j \),

\[
    r_P(x) = \max\{i : p_{ii} \neq 0\}, \quad r_P(y) = \max\{j : p_{jj} \neq 0\}, \quad (2)
\]

and let

\[
    m(P) = r_P(x) + r_P(y).
\]

Clearly, from (1), \( m = m(T) = m(A) + m(C) \). As we are supposing a non-trivial factorization, it follows that \( r_A(x), r_A(y) \leq m(A) < m(T) \). Also useful is the following property.

**Lemma 2.** Let \( M \) be a connected matroid and \( T(M; x, y) \) be its Tutte polynomial with a factorization as in (1). Then the polynomial \( A(x, y) \) also satisfies property 5, that is, if \( r_A(x) \leq i \) or \( r_A(y) \leq j \), then \( a_{ij} = 0 \), except if \( i = r_A(x) \) and \( j = 0 \), or if \( i = 0 \) and \( j = r_A(y) \).

**Proof.** Let \( \alpha = \max\{i : a_{ij} \neq 0 \text{ for some } j\} \) and \( \beta = \max\{j : a_{ij} \neq 0 \} \); define analogously \( \alpha' \) and \( \beta' \) for the polynomial \( C(x, y) \). The monomial \( a_{ij}c_{ii}^a x^{r_A(x)} y^{r_A(y)} \) appears in \( T(M; x, y) \), as it cannot be cancelled, and it is the term with maximum degree of \( x \) in \( T(M; x, y) \). Using property (5) we see that \( \alpha + \alpha' = r(M) \) and \( \beta + \beta' = 0 \), so \( \beta = 0 \) and \( \alpha = r_A(x) \). Thus, the maximum degree of \( x \) in \( A(x, y) \) has coefficient \( a_{r_A(x), 0} \). A similar argument shows that the maximum degree of \( y \) in \( A(x, y) \) has coefficient \( a_{0, r_A(y)} \). 

We next prove two lemmas that together imply Theorem 1.

Let \((A_k), k = 0, 1, \ldots, m(A)\) be the same set of equations as the \((B_k)\), but with the \( a_{st} \) replacing the \( b_{st} \), that is,

\[
    \sum_{s=0}^{k} \sum_{t=0}^{k-s} (-1)^t \binom{k-s}{t} a_{st} = 0. \quad (A_k)
\]

Note that we do not assume \( A(x, y) \) to be the Tutte polynomial of a matroid, hence we do not know whether equations \((A_k)\) hold or not. In fact, we have the following result.

**Lemma 3.** With hypothesis as in Lemma 2, there is at least one equation \((A_l)\) with \( r_A(x) \leq l \leq m(A) \) that does not hold.
Proof. First, for \( r_A(x) \leq k \leq m(A) \) and \( i \geq 0 \) we define the equation \((A_k, i)\) as
\[
\sum_{s=0}^{k} \sum_{t=0}^{k-s} (-1)^{s+i} \binom{k-s}{t} a_{s,t+i} = 0. \tag{A_k, i}
\]
Note that \((A_k, 0)\) is the same equation as \((A_k)\). Now we prove a recurrence relation involving these equations.

Observe that for \( i > 0 \) and \( k > r_A(x) \) the left-hand side of equation \((A_k, i-1)\) is
\[
\sum_{s=0}^{k} \sum_{t=0}^{k-s} (-1)^{s+i-1} \binom{k-s}{t} a_{s,t+i-1}. \tag{3}
\]
Using the fact that \( \binom{k-s}{t} = \binom{k-s-1}{t} + \binom{k-s-1}{t-1} \), and assuming \((a-b) = 0\) for \( a \geq 0, b > 0 \), and also \((a-b) = 0\) if \( a < b \), we can rewrite (3) in the following way:
\[
\sum_{s=0}^{k} \sum_{t=0}^{k-s} (-1)^{s+i-1} \left[ \binom{k-1-s}{t} + \binom{k-1-s}{t-1} \right] a_{s,t+i-1} + (-1)^{i-1} a_{k,i-1}
= \sum_{s=0}^{k-1} \sum_{t=0}^{k-1-s} (-1)^{s+i-1} \binom{k-1-s}{t} a_{s,t+i-1}
+ \sum_{s=0}^{k-1} \sum_{t=0}^{k-1-s} (-1)^{s+i-1} \binom{k-1-s}{t-1} a_{s,t+i-1} + (-1)^{i-1} a_{k,i-1}.
\]
The last term appears because \( \binom{k}{0} \) cannot be decomposed into two binomial coefficients. But by Lemma 2 for this last term we have \( a_{k,i-1} = 0 \), as \( r_A(x) = k \). Also, the first and second terms in the second row of the last expression (after a change of variables) are, respectively, the left-hand side of equations \((A_{k-1,i-1})\) and \((A_{k,i-1})\). So we can write symbolically
\[
(A_{k,i-1}) = (A_{k-1,i}) + (A_{k-1,i-1})
\]
or
\[
(A_{k-1,i}) = (A_{k,i-1}) - (A_{k-1,i-1}) \tag{4}
\]
for \( r_A(x) < k \leq m(A) \) and \( i > 0 \).

Let us suppose now that all equations \((A_k)\) hold for \( r_A(x) \leq k \leq m(A) \) and we will find a contradiction. Consider equation \((A_{r_A(x), r_A(y)})\). By Lemma 2, the only term \( a_{ij} \) involved in this equation that is not zero is
Then the left-hand side of \((A_{r_1(x)} r_1(y))\) reduces to \((r_1^{(x)} a_0, r_1(y)) = a_0, r_1(y)\), which is different from zero. On the other hand, using Eq. (4) repeatedly \(r_1(y)\) times, we can express this nonzero term as a sum of the left-hand sides of equations \((A_{l_0})\) for \(r_1(x) \leq k \leq m(A) = r_1(x) + r_1(y)\), that we are assuming to be all equal to zero. Therefore we obtain a contradiction and we conclude that not all of the \((A_{l_0}) = (A_l)\) hold for \(r_1(x) \leq k \leq m(A)\).

**Lemma 4.** If the coefficients \(a_{ij}\) do not satisfy equation \((A_k)\) for some \(k \leq m(A)\), then \(c_{00} = 0\).

Proof. Let \((A_k)\) be the first equation that does not hold. Equation \((B_k)\) holds because \(k \leq m(A) < m\). First, we rewrite this equation taking into account that

\[
b_{il} = \sum_{h \leq s} \sum_{l \leq t} c_{hl} a_{s-h, t-l}.
\]

Then we have the following equalities for the left-hand side of \((B_k)\).

\[
\sum_{j=0}^{k} \sum_{t=0}^{k-j} (-1)^j \binom{k-s}{t} b_{il} = \sum_{j=0}^{k} \sum_{t=0}^{k-j} (-1)^j \binom{k-s}{t} \sum_{h \leq s} \sum_{l \leq t} c_{hl} a_{s-h, t-l}
\]

\[
= c_{00} \sum_{j=0}^{k} \sum_{t=0}^{k-j} (-1)^j \binom{k-s}{t} a_{st} + \sum_{0 < h \leq l \leq k} c_{hl} \left[ \sum_{s=h}^{k} \sum_{t=l}^{k-s} (-1)^j \binom{k-s}{t} a_{s-h, t-l} \right].
\]

(5)

Note that each \(c_{hl}\) has as coefficient an expression similar to the left hand side of \((A_k)\); in particular, for \(c_{00}\) this coefficient is exactly the left-hand side of equation \((A_k)\). More precisely, we introduce the equation \((A'_n)\).

\[
\sum_{j=0}^{n-i} \sum_{t=0}^{n-i-j} (-1)^{j+i} \binom{n-s}{t+i} a_{st} = 0.
\]

\((A'_n)\)

Observe that the left-hand side of equation \((A'_{k-h, l})\) is the coefficient of \(c_{hl}\) in (5) above: change indices \(s \leftarrow s+h, t \leftarrow t+l\) and note that for \(s > k-l\) and \(t \geq l\) the binomial \(\binom{k-s}{t+i}\) vanishes. Also note that \((A'_0)\) is precisely equation \((A_k)\), which we are assuming holds for \(0 \leq n < k\). Now, we prove that \((A'_n)\) holds for \(1 \leq n \leq k\) and \(1 \leq i \leq n\) using induction on \(n\).

If \(n = 1\), the only possible value for \(i\) is 1 and \((A'_{11})\) reduces to \(a_{00} = 0\), which was supposed from the beginning. Assuming the result for all values
less than \( n \), we use again a formula for the binomial coefficients to decom- pose the left-hand side of equation \( (A'_{n-k}) \) into a sum of previous equations:

\[
\sum_{i=0}^{n-i} \sum_{t=0}^{n-i-t} (-1)^{t+i} \binom{n-s}{t+i} a_{it} = \sum_{i=0}^{n-i} \sum_{t=0}^{n-i-t} (-1)^{t+i} \left[ \binom{n-s-1}{t+i-1} + \binom{n-s-2}{t+i-1} + \cdots + \binom{t+i-1}{t+i-1} \right] a_{it}.
\]

Each binomial coefficient \( \binom{n-s}{t+i} \) is partitioned into exactly \( n-s-t-i+1 \) terms, so that the last expression equals

\[
\sum_{i=0}^{n-1-(i-1)} \sum_{t=0}^{n-1-(i-1)-s} (-1)^{t+i} \binom{n-1-s}{t+i-1} a_{it} + \sum_{i=0}^{n-2-(i-1)} \sum_{t=0}^{n-2-(i-1)-s} (-1)^{t+i} \binom{n-2-s}{t+i-1} a_{it} + \cdots + \sum_{i=0}^{0} \sum_{t=0}^{0} (-1)^{t+i} \binom{1}{t+i-1} a_{it}.
\]

Now it is easy to check that the \( p \)th term in the last sum is equal (up to the sign) to the left hand side of equation \( (A'_{n-k-1,i-1}) \), for \( 1 \leq p \leq n-i+1 \). Thus we obtain the following relation:

\[
(A'_{n-k}) = -((A'_{n-k-1,i-1}) + (A'_{n-k-2,i-1}) + \cdots + (A'_{i-1,i-1})).
\]

If \( i = 1 \), the equations on the right are \( (A'_{n-1}) \), \( (A'_{n-2}) \), \( \ldots \), \( (A'_{0}) \), all of which hold because \( n-1 < k \). If \( i > 1 \), equations \( (A'_{n-k-1,i-1}) \), \( \ldots \), \( (A'_{i-1,i-1}) \) hold by inductive hypothesis. In both cases \( (A'_{n-k}) \) holds, and this concludes the induction.

Using this result we see from (5) that equation \( (B_k) \) reduces to \( c_{00}(A_k) = 0 \). As \( (A_k) \) does not hold, \( c_{00} \) must be zero and the lemma is proved.

The above two lemmas show that \( c_{00} = 0 \) and this establishes the theorem.

**Remark.** The assumption of characteristic zero is necessary, since otherwise property 2 after Lemma 1 does not hold, that is, \( b_{10} \) can be zero because of the characteristic. For example,

\[
T(M(K_4); x, y) = 2x + 2y + 3x^2 + 4xy + 3y^2 + x^3 + y^3 = (x + y)(x + y + x^2 + xy + y^2) \mod 2,
\]

whereas \( M(K_4) \) is a connected matroid.
REFERENCES