Growth constants of minor-closed classes of graphs

Olivier Bernardi, Marc Noy, Dominic Welsh

A minor-closed class of graphs is a set of labelled graphs which is closed under isomorphism and under taking minors. For a minor-closed class \( G \), let \( g_n \) be the number of graphs in \( G \) which have \( n \) vertices. The growth constant of \( G \) is \( \gamma = \limsup (g_n/n!)^{1/n} \). We study the properties of the set \( \Gamma \) of growth constants of minor-closed classes of graphs. Among other results, we show that \( \Gamma \) does not contain any number in the interval \([0, 2]\), besides 0, 1, \( \xi \) and 2, where \( \xi \approx 1.76 \). An infinity of further gaps is found by determining all the possible growth constants between 2 and \( \delta \approx 2.25159 \). Our results give in fact a complete characterization of all the minor-closed classes with growth constant at most \( \delta \) in terms of their excluded minors.

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1. Introduction

Scheinerman and Zito [24] introduced the study of the possible growth rates of hereditary classes of graphs, that is, sets of graphs which are closed under taking induced subgraphs. They provided a broad classification of possible growth rates for hereditary classes, which has been considerably extended in a series of papers by Balogh, Bollobás, and others [2–7]. The problem of determining the possible growth functions for other structures, like permutations, partitions and ordered graphs has also received attention; see [18] and the references therein.

In [8] we studied the growth rates of classes which are minor-closed, that is, closed under deletion and contraction of edges, and deletion of isolated vertices. Clearly, being minor-closed is a much stronger property than being hereditary. Working in this more restricted context, we were able to obtain simpler characterization of the different categories of growth rate and simpler proofs.

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Partly supported by Ministerio de Ciencia e Innovación under grant MTM2008-03020.

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doi:10.1016/j.jctb.2010.03.001
In this paper we focus on growth constants of minor closed-classes. Let \( \mathcal{G} \) be a proper minor-closed class and let \( g_n \) be the number of graphs in \( \mathcal{G} \) with \( n \) vertices. A result by Norine et al. [22], which is implied by a result from Blankenship [9], shows that

\[
g_n \leq c^n n! \tag{1}\]

for some constant \( c \). It follows that \( \gamma = \limsup (g_n/n!)^{1/n} \) is finite. We call it the growth constant of \( \mathcal{G} \) and we write \( \gamma = \gamma(\mathcal{G}) \).

Our goal is to establish properties of the set \( \Gamma \) of growth constants of minor-closed classes of graphs. First note that \( \Gamma \) is countable, as a consequence of the Minor Theorem of Robertson and Seymour [23]. This is not the case for other combinatorial structures, like classes of permutations defined in terms of forbidden patterns, where every real number above a certain value is the growth constant of some class of permutations [25] (see [17,26] for more on growth constants of permutation classes).

In Section 2 we first explore basic properties, like the fact that if \( \gamma \) is in \( \Gamma \), so is \( 2\gamma \). Using several results on graph enumeration we compile a list of interesting growth constants, including the growth constant of planar graphs [15]. We also prove the existence of (infinitely many) limit points in \( \Gamma \). In particular, we show that if the forbidden minors of a class \( \mathcal{G} \) are two-connected but are not cycles, then \( \gamma(\mathcal{G}) \) is a limit point of \( \Gamma \).

In Section 3 we show the existence of gaps in \( \Gamma \). We prove that the only numbers in \( \Gamma \) belonging to the interval \([0, 2]\) are 0, 1, \( \xi \) and 2, where \( \xi \approx 1.76 \) is the inverse of the positive root of \( 1 = z e^z \), and is the growth constant of caterpillars. In fact, we characterize all minor-closed classes whose growth constant is at most 2. For instance, we show that a class \( \mathcal{G} \) has growth constant 0 if and only if \( \mathcal{G} \) does not contain all paths. This is proved using the following dichotomy: either the class \( \mathcal{G} \) contains all paths and its growth constant is at least 1, or else it does not contain some path \( P_k \). In the latter case, graphs in \( \mathcal{G} \) have no simple paths of length \( k \) and we show that the growth constant is equal to 0.

An infinity of further gaps is found in Section 4, by determining all possible growth constants between 2 and \( \delta \approx 2.25159 \), which is the inverse of the positive root of \((z + z^2) \exp(z) = 1\). Here is the statement of the main result (Corollary 4.5).

**Theorem 1.1.** The set of growth constants of minor-closed classes which are below \( \delta \approx 2.25159 \) is \([0, 1, \xi, 2] \cup \Lambda \), where

\[
\Lambda = \bigcup_{k \leq \ell \leq \infty} \{ \lambda(k), \mu(k), \pi(k, \ell) \},
\]

and each of the constants \( \lambda(k), \mu(k), \pi(k, \ell) \) is defined, in Section 4, as the inverse of the smallest positive root of a certain polynomial.

All classes whose growth constant is below \( \delta \) are characterized in terms of whether the class contains or not several particular families of graphs, like paths, caterpillars and related families. A key technical device in the proofs is to consider depth-first search spanning trees and bound the number of pairs \((G, T)\), where \( G \) is a graph in a given class and \( T \) is a DFS spanning tree of \( G \).

We conclude the paper with a brief discussion on unlabeled graphs and some open questions. In particular, we conjecture that for every minor-closed class \( \mathcal{G} \) we have \( \gamma(\mathcal{G}) = \lim n (g_n/n!)^{1/n} \).

We close this introduction with some definitions and basic results. We consider simple labelled graphs. The size of a graph is the number of vertices; graphs of size \( n \) are labelled with \([1, 2, \ldots, n]\). A class of graphs is a family of graphs closed under isomorphism. A class is proper if it does not contain all graphs.

The relation \( H < G \) between graphs means that \( H \) is a minor of \( G \). A class \( \mathcal{G} \) is minor-closed if \( G \in \mathcal{G} \) and \( H < G \) implies \( H \in \mathcal{G} \). A graph \( H \) is a (minimal) excluded minor for a minor-closed class \( \mathcal{G} \) if \( H \notin \mathcal{G} \) but every proper minor of \( H \) is in \( \mathcal{G} \). We write \( \mathcal{G} = \text{Ex}(H_1, H_2, \ldots) \) if \( H_1, H_2, \ldots \) are the excluded minors of \( \mathcal{G} \). By the theory of graph minors developed by Robertson and Seymour [23], the number of excluded minors is always finite.
For a class of graphs $\mathcal{G}$, we let $\mathcal{G}_n$ be the set of graphs in $\mathcal{G}$ with $n$ vertices, and $g_n = |\mathcal{G}_n|$. The (exponential) generating function (GF for short) associated to a class $\mathcal{G}$ is $G(z) = \sum_{n \geq 0} g_n z^n/n!$. Observe that the growth constant of the class $\mathcal{G}$ is the inverse of the radius of convergence $\rho(G)$ of $G(z)$. This is just the definition of the radius of convergence of a power series.

Given two series $A(z) = \sum a_n z^n$ and $B(z) = \sum b_n z^n$ with non-negative coefficients, we say that $B(z)$ dominates $A(z)$ if $a_n \leq b_n$ for $n$ large enough. Clearly in this case we have $\rho(A) \geq \rho(B)$. A useful fact is that for computing the growth constant it is enough to consider connected graphs in $\mathcal{G}$. Indeed, if $c_n$ is the number of connected graphs in $\mathcal{G}$ and $C(z) = \sum c_n z^n/n!$, then $G(z)$ dominates $C(z)$ and is dominated by $\exp C(z)$, which has the same radius of convergence as $C(z)$. Indeed, $\exp(C(x))$ is the GF of the class of graphs whose connected components are in $\mathcal{G}$. Also, if we consider the class $\mathcal{G}'$ of graphs in $\mathcal{G}$ rooted at a vertex, then the growth constant does not change, since $g_n' = ng_n$.

2. The set of growth constants

Recall that the growth constant of a minor-closed class $\mathcal{G}$ is $\gamma(\mathcal{G}) \equiv \limsup(g_n/n!)^{1/n}$. In this section we present some basic properties concerning the set $\Gamma$ of growth constants and then focus on families whose forbidden minors are 2-connected.

Our main interest is to determine which real numbers are growth constants of minor-closed classes of graphs. To begin with, if a class is too small, like the class of graphs with maximum degree one, then the growth constant is 0. The class of graphs whose connected components are paths has growth constant 1; this is because there are $n!$ ways of labelling a path with $n$ vertices (recall that it is enough to consider connected graphs). The class of all forests has growth constant $e$, since the GF $R(z)$ of rooted labelled trees satisfies

$$R(z) = ze^{R(z)},$$

so that the radius of convergence of $R(z)$ is $1/e$. Of course this can be deduced from Cayley’s counting formula $n^{n-2}$ giving the number of labelled trees, but we need (2) later.

A caterpillar is a tree made of a simple path called the spine together with leaves adjacent to the spine. The GF of doubly-rooted caterpillars (caterpillar together with two marked leaves at each extremity of the spine) is $z^2/(1 - ze^2)$. The radius of convergence is the positive root of $1 = ze^2$, which is approximately 0.5671, and its inverse $\xi \approx 1.7632$ is the growth constant of the class consisting of forests of caterpillars. This class plays an important role later.

We also have the following result, based on the so-called apex construction.

Lemma 2.1. If $\gamma$ is in $\Gamma$, then $2\gamma$ is also in $\Gamma$.

Proof. This property follows from an idea by Colin McDiarmid. Suppose $\gamma = \gamma(\mathcal{G})$ and let $\mathcal{A}\mathcal{G}$ be the family of graphs $G$ having a vertex $v$ such that $G - v$ is in $\mathcal{G}$; in this case we say that $v$ is an apex of $G$. It is easy to check that if $\mathcal{G}$ is minor-closed, so is $\mathcal{A}\mathcal{G}$. Now we have

$$2^n|G_n| \leq |\mathcal{A}\mathcal{G}_{n+1}| \leq (n + 1)2^n|G_n|.$$

The lower bound is obtained by taking a graph $G \in \mathcal{G}$ with vertices $[n]$, adding $n + 1$ as a new vertex, and making $n + 1$ adjacent to an arbitrary subset of $[n]$. The upper bound follows the same argument by considering which of the vertices 1, 2, ..., $n + 1$ acts as an apex. Dividing by $n!$ and taking $n$-th roots, we see that $\gamma(\mathcal{A}\mathcal{G}) = 2\gamma(\mathcal{G})$.  

For instance, the class $\mathcal{P}$ of forests of paths has growth constant 1, so that the class $\mathcal{A}\mathcal{P}$ has growth constant 2. This class plays a prominent role later. Several interesting growth constants have been determined in the context of graph enumeration. In Table 1 we show some of them.

We now investigate the topological properties of the set $\Gamma$ and in particular its limit points. We recall that given a set $A$ of real numbers, $a$ is a limit point of $A$ if for every $\epsilon > 0$ there exists $x \in A - \{a\}$ such that $|a - x| < \epsilon$. 

Table 1
A table of some known growth constants.

<table>
<thead>
<tr>
<th>Class of graphs</th>
<th>Growth constant</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ex(P₂)</td>
<td>0</td>
<td>This paper</td>
</tr>
<tr>
<td>Path forests</td>
<td>1</td>
<td>Standard</td>
</tr>
<tr>
<td>Caterpillar forests</td>
<td>$\delta \approx 1.76$</td>
<td>This paper</td>
</tr>
<tr>
<td>Apex-paths</td>
<td>2</td>
<td>This paper</td>
</tr>
<tr>
<td>Forests = Ex(K₃)</td>
<td>$e \approx 2.71$</td>
<td>Standard</td>
</tr>
<tr>
<td>Ex(C₄)</td>
<td>3.63</td>
<td>[16]</td>
</tr>
<tr>
<td>Outerplanar = Ex(K₄, K₂,3)</td>
<td>7.320</td>
<td>[10]</td>
</tr>
<tr>
<td>Ex(K₂,3)</td>
<td>7.327</td>
<td>[10]</td>
</tr>
<tr>
<td>Series parallel = Ex(K₄)</td>
<td>9.07</td>
<td>[10]</td>
</tr>
<tr>
<td>Ex(W₄)</td>
<td>11.54</td>
<td>[16]</td>
</tr>
<tr>
<td>Ex(W₅)</td>
<td>14.67</td>
<td>[16]</td>
</tr>
<tr>
<td>Ex(K₅ − e)</td>
<td>15.65</td>
<td>[16]</td>
</tr>
<tr>
<td>Ex(K₃ × K₂)</td>
<td>16.24</td>
<td>[16]</td>
</tr>
<tr>
<td>Planar</td>
<td>27.227</td>
<td>[15]</td>
</tr>
<tr>
<td>Embeddable in a fixed surface</td>
<td>27.227</td>
<td>[19]</td>
</tr>
<tr>
<td>Ex(K₃,3)</td>
<td>27.2293</td>
<td>[14]</td>
</tr>
<tr>
<td>Ex(K₃,3)</td>
<td>27.2295</td>
<td>[14]</td>
</tr>
</tbody>
</table>

**Theorem 2.2.** Let $H_1, \ldots, H_k$ be 2-connected graphs which are not cycles. Then the growth constant $\gamma = \gamma(\text{Ex}(H_1, \ldots, H_k))$ is a limit point of $\Gamma$.

The proof of Theorem 2.2 uses results from [21] about so-called addable classes of graphs. Based on these results it was shown in [8], that if the excluded minors of a minor-closed class $\mathcal{G}$ are all 2-connected, then the sequence $(g_n/n!)^{1/n}$ converges toward $\gamma(\mathcal{G})$. A stronger result was obtained by McDiarmid in [20]: under the same conditions, the sequence $g_n/(ng_{n-1})$ converges toward $\gamma(\mathcal{G})$.

**Lemma 2.3.** Let $H_1, H_2, \ldots, H_k$ be a family of 2-connected graphs, and let $\mathcal{H} = \text{Ex}(H_1, \ldots, H_k)$. If $G$ is a 2-connected graph in $\mathcal{H}$ and $\mathcal{G} = \mathcal{H} \cap \text{Ex}(G)$, then $\gamma(\mathcal{G}) < \gamma(\mathcal{H})$.

**Proof.** By Theorem 4.1 from [21], the probability that a random graph in $\mathcal{H}_n$ does not contain $G$ as a subgraph is at most $e^{-\alpha n}$ for some $\alpha > 0$. Hence, the probability that a random graph in $\mathcal{H}_n$ does not contain $G$ as a minor is at most $e^{-\alpha n}$, that is,

$$\frac{|\mathcal{G}_n|}{|\mathcal{H}_n|} \leq e^{-\alpha n}.
$$

The condition on 2-connectivity guarantees that the growth constants involved are in fact limits. Taking limits, the former inequality implies

$$\frac{\gamma(\mathcal{G})}{\gamma(\mathcal{H})} = \lim \frac{|\mathcal{G}_n|^{1/n} n!^{1/n}}{|\mathcal{H}_n|} \leq \lim (e^{-\alpha n})^{1/n} = e^{-\alpha} < 1. \quad \square
$$

**Proof of Theorem 2.2.** For $k \geq 3$, let $G_k = \mathcal{G} \cap \text{Ex}(C_k)$, where $C_k$ is the cycle of size $k$, and let $\gamma_k = \gamma(G_k)$. Because of Lemma 2.3 the $\gamma_k$ are strictly increasing and $\gamma_k < \gamma$ for all $k$. It follows that $\gamma' = \lim_{k \to \infty} \gamma_k$ exists and $\gamma' \leq \gamma$. In order to show equality we proceed as follows.

Let $g_n = |\mathcal{G}_n|$, $g_{k,n} = |\mathcal{G}_{k,n}|$, $f_n = g_n/(e^2n!)$, and $f_{k,n} = g_{k,n}/(e^2n!)$. Using the fact that the excluded minors of $G_k$ are 2-connected, it follows from the proof of Theorem 3.3 in [21] that the sequence $(f_{k,n})_{n \in \mathbb{N}}$ is supermultiplicative, and $\gamma_k = \lim_{n \to \infty} (f_{k,n})^{1/n} = \sup_n (f_{k,n})^{1/n}$.

By definition $\gamma = \lim \sup_{n \to \infty} (g_n/n!)^{1/n} = \lim \sup_{n \to \infty} (f_n)^{1/n}$. Hence, for every $\epsilon > 0$ there exists $N$ such that

$$(f_N)^{1/N} \geq \gamma - \epsilon.$$
Moreover, since a graph on less than \( k \) vertices cannot contain \( C_k \) as a minor, we have \( g_{k,N} = g_N \) for \( k > N \). Equivalently, \( f_{k,N} = f_N \) for \( k > N \). Thus, for all \( k > N \),
\[
\gamma_k \geq (f_{k,N})^{1/N} = (f_N)^{1/N} \geq \gamma - \epsilon.
\]
This implies \( \gamma' = \lim \gamma_k \geq \gamma \). \( \square \)

Notice that Theorem 2.2 applies to all the classes in Table 1 starting at the class of outerplanar graphs. However, it does not apply to the classes of forests. In this case a direct proof based on generating functions shows that \( e \) is a limit point of \( \Gamma \). In Section 4 we show that there are an infinite number of limit points below \( e \), and that there is point which is a limit point of limit points.

**Remark.** All our examples of limit points in \( \Gamma \) come from strictly increasing sequences of growth constants that converge to another growth constant. The question arises whether it is possible to have an infinite strictly decreasing sequences of growth constants? Assuming the set of minor-closed classes of graphs is well quasi-ordered by inclusion (this is so far an open problem [12]), an infinite decreasing sequence \( \gamma_1 > \gamma_2 > \cdots \) of growth constants cannot exist. For consider the corresponding sequence of graph classes \( \Gamma_1, \Gamma_2, \ldots \). For some \( i < j \) we must have \( \Gamma_i \subseteq \Gamma_j \), but this implies \( \gamma_i \leq \gamma_j \).

### 3. Growth constants between 0 and 2

In this section we characterize the minor-closed classes of graphs which have a growth constant at most 2. We begin with the classes having growth constant 0.

Let us recall that the growth constant of a class of labelled graphs \( \Gamma \) is the inverse of the radius of convergence of the associated generating function \( G(z) \), and it is 0 if \( G(z) \) is analytic everywhere. Recall also that for a minor-closed class of graphs \( \Gamma \), the growth constant of \( \Gamma \) is equal to the growth constant of the subclass of connected graphs in \( \Gamma \).

**Theorem 3.1.** A minor-closed class has growth constant 0 if and only if it does not contain all paths.

Since there are \( n! \) ways of labelling a path of size \( n \), a class \( \Gamma \) containing all paths has growth constant at least 1. Thus, it suffices to prove that if \( \Gamma \) does not contain all paths then its growth constant is 0. We prove this property by using DFS-trees.

A **DFS-tree** of a connected graph \( G \) is a rooted spanning tree \( T \) obtained by a depth-first search algorithm on \( G \) (see, for instance, [11]). A **DFS-rooted graph** is a pair \((G, T)\) consisting of a connected graph \( G \) together with a DFS-tree \( T \). For technical reasons, we allow DFS-rooted graphs \((G, T)\) to contain some double-edges \( e, f \) if \( e \) is in \( T \) and \( f \) is not in \( T \). Some DFS-rooted graphs are represented in Fig. 7. The following property is well known.

**Lemma 3.2 (Folklore).** A spanning tree \( T \) of a graph \( G \) is a DFS-tree if and only if every edge of \( G \) which is not in \( T \) joins a vertex of \( G \) to one of its ancestors in \( T \).

Recall that the **height** of a rooted tree is the length of the longest path from the root to a leaf.

**Lemma 3.3.** For fixed \( k \), let \( d_{k,n} \) be the number of DFS-rooted graphs \((G, T)\) such that \( G \) has \( n \) vertices and \( T \) has height at most \( k \). Then the generating function \( D_k(z) = \sum_{n \geq 0} d_{k,n} z^n/n! \) has infinite radius of convergence.

**Proof.** For a fixed integer \( k \), the generating function \( T_k(z) \) of the class \( T_k \) of rooted labelled trees of height at most \( k \) has infinite radius of convergence; see [13, Section III.8.2] for an explicit expression of \( T_k(z) \). A DFS-rooted graph \((G, T)\) such that \( T \) has height at most \( k \) is obtained by choosing a tree \( T \) in \( T_k \) and adding some **external edges** (edges which are in \( G \) but not in \( T \)). By Lemma 3.2, this can be done by choosing for each vertex of \( T \) a subset of its ancestors. Since there are at most \( k \) ancestors, we have at most \( 2^k \) possibilities for each vertex. This proves that the exponential generating function
D_k(z) is dominated by the series T_k(2^k z). This concludes the proof since T_k(2^k z) has infinite radius of convergence. □

**Proof of Theorem 3.1.** Let $\mathcal{G}$ be a minor-closed class of graphs not containing the path $P_k$. For any connected graph $G$ in $\mathcal{G}$, choosing a DFS-tree $T$ gives a DFS-rooted graph $(G, T)$ such that $T$ has height at most $k$ (since $G$ does not contain $P_k$ as a minor). Therefore, the GF of the class of connected graphs in $\mathcal{G}$ is dominated by $D_k(z)$, and by Lemma 3.3 it has infinite radius of convergence. This implies that $\mathcal{G}$ has growth constant 0. □

**Remark.** The proof of Theorem 3.1 uses only containment under subgraphs, hence the result holds more generally for monotone classes of graphs.

Next we characterize the classes that have growth constant between 0 and 2. First we define some families of graphs represented in Fig. 1. Recall that a caterpillar is a tree made of a simple path called the spine together with leaves adjacent to the spine. A double-caterpillar is a tree obtained from a caterpillar by attaching at most one leaf to each leaf of the caterpillar. A thick-caterpillar is a graph obtained from a caterpillar by attaching a triangle to some edges of the spine. An apex-path is made of a simple path together with an extra vertex connected to some vertices of the path.

The main result in this section is the following.

**Theorem 3.4.** Let $\mathcal{G}$ be a minor-closed class of graphs.

(i) The growth constant of $\mathcal{G}$ is 1 if and only if $\mathcal{G}$ contains all paths but neither all caterpillars nor all apex-paths.

(ii) The growth constant of $\mathcal{G}$ is $\xi \approx 1.76$ if and only if $\mathcal{G}$ contains all caterpillars but neither all double-caterpillars nor all thick-caterpillars nor all apex-paths.

**Example.** Theorem 3.4 implies, for instance, that the class of graphs not containing a given star (which is not a path) as a minor has growth constant 1. The set of graphs not containing a given cycle nor a given double-caterpillar (which is not a caterpillar) has growth constant $\xi$.

Recall that the growth constant of caterpillars is $\xi \approx 1.76$, and that the growth constant of apex-paths is 2. The growth constant of the class of double-caterpillars is the inverse of the positive solution of $z \exp(z + z^2) = 1$, which is approximately 2.06, while for thick-caterpillars it is the inverse of the positive solution of $(z + z^2) \exp(z) = 1$, which is approximately $\delta \approx 2.25$. They are both above 2, so that Theorem 3.4 immediately gives the following corollary.

**Corollary 3.5.** The only growth constants of minor-closed classes of graphs less than 2 are 0, 1 and $\xi$.

The rest of this section is devoted to the proof of Theorem 3.4. Observe first that by Theorem 3.1 a minor-closed class of graph $\mathcal{G}$ having growth constant 1 must contain all paths. Moreover it cannot contain all caterpillars nor all apex-paths. Hence, in order to prove (i) it suffices to prove that a minor-closed class $\mathcal{G}$ containing neither all caterpillars nor all apex-paths has growth constant at most 1. Similarly, in order to prove (ii) it suffices to prove that a minor-closed class $\mathcal{G}$ containing neither all double-caterpillars nor all thick-caterpillars nor all apex-paths has growth constant at most $\xi$. We explain briefly our strategy to obtain these upper bounds.
If a minor-closed class does not contain all caterpillars (respectively, double-caterpillars, thick-caterpillars, apex-paths), then it does not contain $C_k$ (respectively, $E_k$, $D_k$, $A_k$) for some $k$, where these graphs are depicted in Fig. 2. Hence, if a minor-closed class $G$ contains neither all caterpillars nor all apex-paths, then $G$ is contained in $Ex(A_k, A_k)$ for some $k$. Similarly if a minor-closed class $G$ contains neither all double-caterpillars nor all thick-caterpillar nor all apex-paths, then $G$ is contained in $Ex(A_k, D_k, E_k)$ for some $k$.

From the preceding remarks, proving property (i) of Theorem 3.4 amounts to proving that for every $k$ the growth constant of $Ex(A_k, C_k)$ is at most 1. To this end it is enough to prove that the GF $G_k(z)$ of DFS-rooted graphs $(G, T)$ with $G \in Ex(A_k, C_k)$ has radius of convergence at least 1. We will prove that $G_k(z)$ is dominated by $D_{f_1(k)}(P(z))$ for some function $f_1(k)$, where the series $D_{f_1(k)}(z)$ is as in Lemma 3.3 and $P(z)$ has radius of convergence 1. Since $D_{f_1(k)}(z)$ has infinite radius of convergence, $G_k(z)$ has radius of convergence at least 1. Similarly, proving (ii) amounts to proving that the growth constant of $Ex(A_k, D_k, E_k)$ is at most $\xi$.

**Definition.** Let $(G, T)$ be a DFS-rooted graph and let $r$ be the root of $T$.

A $T$-path is a path $v_0, v_1, \ldots, v_{k+1}$ of $T$ not containing $r$ such that for $i = 1, \ldots, k$ the vertex $v_i$ is adjacent in $G$ only to $v_{i-1}$ and $v_{i+1}$. The path-reduction $(\bar{G}, \bar{T})$ of $(G, T)$ is obtained by replacing every maximal $T$-path $v_0, v_1, \ldots, v_{k+1}$ by the edge $(v_0, v_{k+1})$ (which is considered as an edge of $\bar{T}$).

A $T$-caterpillar is a path $v_0, v_1, \ldots, v_{k+1}$ of $T$ not containing $r$ such that for $i = 1, \ldots, k$ the vertex $v_i$ is adjacent in $G$ only to $v_{i-1}$ and $v_{i+1}$, and possibly to vertices of degree one distinct from the root. The caterpillar-reduction $(\bar{G}, \bar{T})$ of $(G, T)$ is obtained by replacing every maximal $T$-caterpillar $v_0, v_1, \ldots, v_{k+1}$ by the edge $(v_0, v_{k+1})$ (which is considered as an edge of $\bar{T}$).

It is clear from the characterization of DFS-trees given in Lemma 3.2 that the path-reduction and the caterpillar-reduction of a DFS-rooted graph are both DFS-rooted graphs.

Given a graph $G$ and a path $P$ of $G$, a $P$-bridge is an edge not in $P$ with both endpoints in $P$.

**Lemma 3.6.** There exists a function $f_0(k)$ such that if $P$ is a path in a graph $G$ not containing $D_k$ as a minor, then there is no set of $f_0(k)$ $P$-bridges whose endpoints are all distinct.

**Proof.** We prove the property for $f_0(k) = 16k^4$. Suppose that $E$ is a set of $16k^4$ $P$-bridges whose endpoints are all distinct. We say that two $P$-bridges $e, e' \in E$ cross if their endpoints alternate along the path $P$ (Fig. 3(a)).

**Claim.** There are less than $8k^2$ $P$-bridges in $E$ crossing a given $P$-bride $e \in E$.

**Proof.** Suppose there is a subset $E'$ of $E$ of $8k^2$ $P$-bridges crossing the $P$-bridge $e = (u, v)$. We choose an orientation of the path $P$ and say that a vertex of $P$ appears before or after another vertex $v'$ along $P$ according to this orientation. By symmetry, we can assume there is a subset $\{e_1, \ldots, e_c\} \subseteq E'$ of $c = 4k^2$ $P$-bridges all of them having an endpoint after $u$ and $v$ along $P$. Let $u_i$ be the endpoint of $e_i$ between $u$ and $v$, with the convention that $u_1, u_2, \ldots, u_c$ appear in this order along $P$, and let $v_i$ be the endpoint of $e_i$ after $u$ and $v$. Then there exists a subset of $2k = \sqrt{c}$ indices $1 \leq i_1 < i_2 < \cdots <
There are at least \( k \) edges \( f_1, \ldots, f_k \) in \( F \) all having the same height. Now the minor \( D_k \) is contained in the subgraph of \( G \) made of \( P \) together with the edges \( f_1, \ldots, f_k \), and we reach a contradiction. \( \square \)

Given a path \( P \) of \( G \), the \( P \)-degree of a vertex \( v \) is the number of neighbors of \( v \) in \( P \).

**Lemma 3.7.** Let \( P \) be a path of a graph \( G \) not containing \( D_k \) as a minor. If the \( P \)-degree of every vertex is less than \( d \), then the number of vertices of \( P \) incident to \( P \)-bridges is less than \( 2df_0(k) \).

**Proof.** Suppose there are \( n = 2df_0(k) \) vertices \( v_1, \ldots, v_n \) joined, respectively, to \( w_1, \ldots, w_n \) by the \( P \)-bridges \( e_1, \ldots, e_n \). Since the \( P \)-degree of any vertex is less than \( d \), there are at least \( n/d = 2f_0(k) \) distinct vertices among \( w_1, \ldots, w_n \). Hence, there are at least \( f_0(k) \) \( P \)-bridges with distinct endpoints among \( e_1, \ldots, e_n \). This contradicts Lemma 3.6. \( \square \)

The following key lemmas bound the height of DFS-trees after path- and caterpillar-reductions.

**Lemma 3.8.** There exists a function \( f_1(k) \) such that if a connected graph \( G \) does not contain neither \( A_k \) or \( C_k \) as minor, and \( T \) is a DFS-tree of \( G \), then the path-reduction \((\hat{G}, \hat{T})\) of \((G, T)\) has height at most \( f_1(k) \).

**Proof.** We prove the claim for \( f_1(k) = 4kf_0(k) + k^2 \). It suffices to prove that no simple path \( P \) of \( G \) contains more than \( f_1(k) \) vertices of degree more than two.

Let \( P \) be a simple path of \( G \). First remark that \( G \) does not contain \( D_k \) as a minor since it does not contain \( C_k \). Moreover, the \( P \)-degree of any vertex is less than \( 2k \), otherwise \( G \) would contain \( A_k \). Thus Lemma 3.7 ensures that less than \( 4kf_0(k) \) vertices of \( P \) are incident to \( P \)-bridges. Therefore it suffices to show that there are less than \( k^2 \) vertices of \( P \) adjacent to a vertex not in \( P \).
Suppose there are \( n = k^2 \) distinct vertices \( u_1, \ldots, u_n \) of \( P \) joined respectively to vertices \( v_1, \ldots, v_n \) not in \( P \) by the edges \( e_1, \ldots, e_n \). Since \( G \) does not contain \( A_k \) as a minor, there are at least \( n/k = k \) distinct vertices among \( v_1, \ldots, v_n \). Thus, the minor \( C_k \) is contained in the subgraph of \( G \) made of \( P \) together with the edges \( e_1, \ldots, e_n \). We reach a contradiction. □

**Lemma 3.9.** There exists a function \( f_2(k) \) such that if a connected graph \( G \) does not contain any of \( A_k, D_k, E_k \) as a minor, and \( T \) is a DFS-tree of \( G \), then the caterpillar-reduction \((\hat{G}, \hat{T})\) of \((G, T)\) has height at most \( f_2(k) \).

**Proof.** We prove the claim for \( f_2(k) = 4k f_0(k) + 4k^2 (f_0(k) + k) \). It is enough to prove that no simple path \( P \) of \( G \) contains more than \( f_2(k) \) vertices which are either incident to a \( P \)-bridge or adjacent to a vertex not in \( P \) which is not of degree one.

Let \( P \) be a simple path of \( G \). The \( P \)-degree of any vertex is less than \( 2k \), otherwise \( G \) contains \( A_k \). Thus Lemma 3.7 ensures that less than \( 4k f_0(k) \) vertices of \( P \) are incident to a \( P \)-bridge. Therefore it suffices to show that there are less than \( 4k^2 (f_0(k) + k) \) vertices of \( P \) adjacent to a vertex not in \( P \) which is not of degree one.

Suppose there are \( n = 4k^2 (f_0(k) + k) \) vertices \( u_1, \ldots, u_n \) joined to some vertices \( v_1, \ldots, v_n \) not in \( P \) which are not of degree one. Let \( e_i = (u_i, v_i) \), and for \( 1 \leq i \leq n \) choose a vertex \( w_i \neq u_i \) joined to \( v_i \) by an edge \( f_i \). Since \( G \) does not contain \( A_k \), there is a set \( I \subseteq \{1, \ldots, n\} \) of size \( |I| = n/k = 4k (f_0(k) + k) \) such that \( v_i \neq v_j \) for \( i \neq j \) in \( I \). There is also a subset \( J \subseteq I \) of size \( |J| = |I|/2k = 2 (f_0(k) + k) \) such that \( w_i \neq w_j \) for any pair \( i \neq j \) in \( J \) (otherwise the vertices \( w_i \) coincide for \( 2k \) distinct integers \( i \) in \( I \) and the minor \( A_k \) appears). Finally, one can find a subset \( L \subseteq J \) of size \( |L| = |J|/2 = f_0(k) + k \) such that the vertices \( u_i, v_i, w_i, w_j \) are all distinct for \( i \neq j \) in \( L \). Since \( E_k \) is not a minor of \( G \), there are less than \( k \) indices \( i \) in \( L \) such that \( w_i \) is not in \( P \). Hence, there must be a subset \( M \subseteq L \) of size \( f_0(k) \) such that \( w_i \) is in \( P \) for all \( i \) in \( M \). In this case, contracting the edges \( (v_i, w_i) \) for all \( i \) in \( M \) gives a set of \( f_0(k) \) \( P \)-bridges with endpoints all disjoint. This contradicts Lemma 3.6. □

**Proof of Theorem 3.4.** By Lemma 3.8, a DFS-rooted graph \((G, T)\) such that \( G \) is in \( \text{Ex}(A_k, C_k) \) is obtained from a DFS-rooted graph \((\hat{G}, \hat{T})\) with \( \hat{T} \) of height at most \( f_1(k) \) by replacing every edge of \( \hat{T} \) by a path. The number of edges in \( \hat{T} \) is the number of vertices minus one, hence the generating function of DFS-rooted graphs \((G, T)\) such that \( G \) is in \( \text{Ex}(A_k, C_k) \) is dominated by \( \sum d_{f_1(k),n} P(z)^n z^n/n! = D_{f_1(k)}(zP(z)) \), where \( P(z) = 1/(1-z) \) is the generating function of paths rooted at an endpoint. By the preceding remarks, this implies property (i).

Similarly, Lemma 3.9 implies that a DFS-rooted graph \((G, T)\) with \( G \) in \( \text{Ex}(A_k, D_k, E_k) \) is obtained from a DFS-rooted graph \((\hat{G}, \hat{T})\) with \( \hat{T} \) of height at most \( f_2(k) \) by replacing every edge of \( \hat{T} \) by a caterpillar. Hence, the generating function of DFS-rooted graphs \((G, T)\) such that \( G \) is in \( \text{Ex}(A_k, D_k, E_k) \) is dominated by \( D_{f_2(k)}(zC(z)) \), where \( C(z) = z/(1-z \exp(z)) \) is the generating function of (doubly rooted) caterpillars. Since \( C(z) \) has radius of convergence \( 1/\xi \), this implies property (ii). □

**4. Growth constants between 2 and 2.25159**

In this section we characterize all minor-closed classes of graphs which have a growth constant between 2 and \( \delta \approx 2.25159 \). Recall that \( \delta \) is the inverse of the least positive root of \( z + z^2 \exp(z) = 1 \) and is the growth constant of thick-caterpillars. There is an infinity of distinct growth constants between 2 and \( \delta \) and our first task is to list them all.

Recall that a rooted graph is a graph with a distinguished vertex \( r \) called the root. A rooted graph \( H \) is a rooted-minor of another rooted graph \( G \) if \( H \) is obtained from \( G \) by a series of deletions and contractions, where the contraction of an edge \( e \) incident to \( r \) gives rise to the new root. The root-components of a connected rooted graph \( G \) are the connected rooted graphs obtained by splitting the root \( r \); that is, each root-component is obtained by adding \( r \) as a root to a connected component \( C \) of \( G \setminus r \); plus all the edges joining \( r \) to \( C \). Observe that there are as many root-components as blocks (2-connected components) containing \( r \).

We now define several minor-closed classes of rooted graphs which play an important role in the sequel. Let \( L_k, M_k \) and \( P_{k,\ell} \) be as in Fig. 4. By a special convention, \( P_{0,\ell} \) is the star with \( \ell + 1 \) leaves,
one of which is the root. For all $k$, $\ell$ we denote by $L_k$, $M_k$, and $P_{k,\ell}$ the class of connected rooted graphs such that all the root-components are rooted-minors, respectively, of $L_k$, $M_k$, and $P_{k,\ell}$. Observe that neither $M_k$ nor $P_{k,\ell}$ can be a root-component of a graph (since they have two root-components); but some of their rooted-minors can be root-components. Observe that $P_{k,\ell} = P_{k,k}$ whenever $k \geq \ell$, hence we can restrict our attention to the case $k \leq \ell$. We also define

$$L_\infty = \bigcup_k L_k, \quad M_\infty = \bigcup_k M_k, \quad P_{k,\infty} = \bigcup_\ell P_{k,\ell}, \quad P_{\infty,\infty} = \bigcup_k P_{k,\infty}.$$  

Finally we set

$$S = L_\infty \cup M_\infty \cup P_{\infty,\infty}. \quad (3)$$

The class $S$ is minor-closed and in the next lemma we find its forbidden minors (refer to Fig. 4).

**Lemma 4.1.** The set $S$ consists of the connected rooted graphs containing none of the $Z_i$, $i = 1, \ldots, 6$, as a rooted-minor.

**Proof.** The graphs in $S = L_\infty \cup M_\infty \cup P_{\infty,\infty}$ do not contain any of the $Z_i$, $i = 1, \ldots, 6$, as a rooted-minor. Indeed, $Z_1$ cannot appear as a minor because the path of length three cannot arise by contracting edges in $M_k$; a similar argument applies to the remaining graphs. Suppose conversely that a connected graph $G$ contains none of the $Z_i$ as root-minors. We have two cases.

If $G$ contains cycles, they must be triangles which are root-components; otherwise $G$ would contain $Z_5$ or $Z_6$. These triangles are rooted-minors of $M_k$ for each $k$. The remaining root-components of $G$ must be trees of height at most two with a root of degree one, otherwise $G$ would contain $Z_1$. These trees are rooted-minors of $M_k$ for some $k$, hence $G$ belongs to $M_\infty$.

If $G$ has no cycles, then all its root-components are rooted-trees such that no vertex of degree greater than two has a grand-child, since otherwise $G$ would contain $Z_3$. If none of the root-components has a vertex of degree greater than two, then $G$ is in $L_\infty$. Otherwise, all its root-components are of height at most three, since otherwise $G$ would contain $Z_2$ or $Z_4$. Hence, these root-components are rooted-minors of $P_{k,\ell}$ for some $k, l$, and $G$ is in $P_{\infty,\infty}$.

**Definition.** If $G$ is a rooted-graph, we denote by $G^d$ the graph obtained by taking $d$ copies of $G$ and joining their root-vertices along a path. Given a class $C$ of rooted graphs, we let $\text{Cat}(C)$ be the graphs obtained by taking graphs in $C$ and joining their roots along a path (see Fig. 5). We call them $C$-caterpillars.

We now define a set $\Lambda$ of constants which will be proved to be the set of all growth constants between 2 and $\delta \approx 2.25159$. In the following table, for $k \leq \ell \leq \infty$, the constants $\lambda(k)$, $\mu(k)$ and $\pi(k,\ell)$ are defined as the inverses of the least positive root of the equation to their right, where $\exp^{\leq k}(z) = \sum_{i=0}^{k} z^i/i!$ is the truncated exponential and $\exp^{\leq \infty}(z) = \exp(z)$. The constants and the corresponding equations are:

---

Fig. 4. Families of rooted graphs.

Fig. 5. A $C$-caterpillar obtained by joining the graphs $G_1, \ldots, G_d \in C$ along a path.
We let \( \Lambda \) be the set of all the previous constants:

\[
\Lambda = \bigcup_{k \leq \ell \leq \infty} \{ \lambda(k), \mu(k), \pi(k, \ell) \}.
\]

**Lemma 4.2.** For all \( k \leq \ell \leq \infty \) the growth constants of the classes of \( L_k \)-caterpillars, \( M_k \)-caterpillars, and \( P_{k, \ell} \)-caterpillars are, respectively, \( \lambda(k) \), \( \mu(k) \) and \( \pi(k, \ell) \). Moreover they are all distinct and less than \( \delta \approx 2.25159 \).

**Proof.** The fact that the constants are all distinct is an easy consequence of Lindemann’s theorem saying that if \( \alpha \neq 0 \) is algebraic, then \( e^{\alpha} \) is transcendental [1]. To see that the constants are less than \( \delta \) it suffices to check that \( \lambda(\infty) \approx 2.24 \), \( \mu(\infty) \approx 2.23 \) and \( \pi(\infty, \infty) \approx 2.243 \).

The fact that the growth constants are as stated is a consequence of basic properties of generating functions. Let us work out in detail the case of \( \lambda(k) \). A rooted graph in \( L_k \) is a rooted tree all of whose root-components are paths of length at most \( k \). The associated GF is then \( L_k(z) = z \exp(z + z^2 + \cdots + z^k) = \exp(z(1 - z^k)/(1 - z)) \); the factor \( z \) encodes the root, the exponential encodes the unordered collection of root-components, and the term \( z + \cdots + z^k \) encodes a path of length at most \( k \). Now the class of doubly-rooted \( L_k \)-caterpillar (obtained by marking two vertices at the beginning and end of the spine) is an ordered sequence of graphs in \( L_k \); its GF is then \( 1/(1 - L_k(z)) \). The radius of convergence of \( 1/(1 - L_k(z)) \) is the least positive root of \( L_k(z) = 1 \) (since this series as non-negative coefficients), that is, \( 1/\lambda(k) \). Therefore, the class of doubly-rooted \( L_k \)-caterpillars has growth constant \( \lambda(k) \) and the same is true for the class of unrooted \( L_k \)-caterpillars (since rootings do not change the growth constant).

For the remaining families the argument is similar, since the GF of \( L_\infty \), \( M_k \) and \( P_{k, \ell} \) are, respectively, \( z \exp(z/(1 - z)) \), \( z \exp(z \exp \leq k(z) + z^2/2) \) and \( z \exp(z \exp \leq \ell(z) + z^2 \exp \leq k(z) - z^2) \).

Observe that the class of graphs whose connected components are \( L_k \)-caterpillars (resp. \( M_k \)-caterpillars, \( P_{k, \ell} \)-caterpillars) is minor-closed. We obtain the following.

**Corollary 4.3.** Every number in \( \Lambda \) is the growth constant of some minor-closed class of graphs. Moreover, for \( k \leq \ell \leq \infty \), if a class \( G \) contains \( L_k^d \) (resp. \( M_k^d \), \( P_k^d \)) for all integers \( d \), then its growth constant is at least \( \lambda(k) \) (resp. \( \mu(k) \), \( \pi(k, \ell) \)).

Recall that \( G^d \) is constructed by joining \( d \) rooted copies of \( G \) along a path. Let \( G \) be a minor-closed class of graphs. We denote by \( \ell(G) \) (resp. \( m(G) \), \( p(G) \)) the largest integer \( n \) such that \( L_n^d \) (resp. \( M_n^d \), \( P_n^d \)) belongs to \( G \) for all \( d \). By convention the maximum of the empty set is \( - \infty \) and the maximum of \( \mathbb{N} \) is \( + \infty \). For all \( k \in \mathbb{N} \), we also denote by \( p_k(G) \) the largest \( n \) such that \( P_{k,n}^d \) belongs to \( G \) for all \( d \).

**Theorem 4.4.** Let \( G \) be a minor-closed class of graphs. The growth constant \( \gamma(G) \) of \( G \) is less than \( \delta \) if and only if there exists \( k \) such that none of the graphs \( B_k, B_k', D_k, Z_k^1, Z_k^2, Z_k^3 \) (see Figs. 2 and 4) is in \( G \). In this case, \( \gamma(G) \) is determined as follows:

1. If \( G \) does not contain \( E_k \) for all \( k \) (equivalently \( p(G) \) and \( p_0(G) \) are non-positive), then \( \gamma(G) \) is at most 2.
2. If \( p(G) = + \infty \), then \( \gamma(G) = \pi(\infty, \infty) \).
3. Otherwise, \( \gamma(G) \) is the maximum of \( \lambda(\ell(G)), \mu(m(G)) \) and \( \pi(k, p_k(G)) \) for \( k = 0, \ldots, p(G) \).
Example. Theorems 3.4 and 4.4 characterize the minor-closed classes having growth constant below \(\delta \approx 2.25159\). For instance, the class of graphs whose excluded minor is a pair of disjoint stars has growth constant 2.

From Theorem 4.4 one obtains immediately the following corollary.

Corollary 4.5. The set of growth constants of minor-closed classes which are below \(\delta \approx 2.25159\) is \([0, 1, \xi, 2] \cup \Lambda\).

The rest of this section is devoted to the proof of Theorem 4.4. For the convenience of the reader, we break the proof into several lemmas; the final proof comes at the end of the section. Let us start with the easy part.

Lemma 4.6. Let \(G\) be a minor closed class of graphs.

1. If \(G\) contains \(D_k\) for all \(k\), then its growth constant is at least \(\delta \approx 2.25159\).
2. If \(G\) contains \(B_k\) for all \(k\), then its growth constant is at least the inverse \(\omega_1 \approx 2.84\) of the smallest positive root of \(2 \exp(z) = 1\).
3. If \(G\) contains \(B'_k\) for all \(k\), then its growth constant is at least \(\omega_2 = 4\).
4. If \(G\) contains \(Z^1_k\) for all \(k\), then its growth constant is at least the inverse \(\omega_3 \approx 2.27\) of the smallest positive root of \(z \exp(z + 3z^2/2 + z^3) = 1\).
5. If \(G\) contains \(Z^2_k\) for all \(k\), then its growth constant is at least the inverse \(\omega_4 \approx 2.25165\) of the smallest positive root of \(z \exp(z + z^3 + 3z^2/2 + z^4) = 1\).
6. If \(G\) contains \(Z^3_k\) for all \(k\), then its growth constant is at least \(\omega_4\).

Moreover, the constants \(w_1, w_2, w_3, w_4\) are all above \(\delta\).

The proof of Lemma 4.6 relies on easy generating function techniques and is omitted. Now, any minor-closed class \(G\) having growth constant less than \(\delta\) is included in \(\text{Ex}(B_k, B'_k, D_k, Z^1_k, Z^2_k, Z^3_k)\) for some \(k\). Moreover, by Corollary 4.3, the growth constant of a minor-closed class is at least \(\pi(\omega, \infty)\) if \(p(G) = +\infty\), and at least the maximum of \(\lambda(\ell(G)), \mu(m(G))\) and \(\pi(k, p_k(G))\) for \(k = 0, \ldots, \omega(G)\) otherwise. In particular, it can be checked that this lower-bound is below 2 if and only if \(p(G)\) and \(p_0(G)\) are non-positive (equivalently, \(G\) does not contain \(E_k\) for all \(k\)). Thus, proving Theorem 4.4 amounts to proving the following: if \(G\) is a minor-closed class contained in \(\text{Ex}(B_k, B'_k, D_k, Z^1_k, Z^2_k, Z^3_k)\), then its growth constant is at most \(\pi(\omega, \infty)\) if \(p(G) = \infty\), and at most the maximum of \(\omega, \lambda(\ell(G)), \mu(m(G))\) and \(\pi(k, p_k(G))\) for \(k = 0, \ldots, \omega(G)\) otherwise.

The strategy is similar to the one in the previous section. For a minor-closed class \(G\) contained in \(\text{Ex}(B_k, B'_k, D_k, Z^1_k, Z^2_k, Z^3_k)\) we consider the generating function \(H_G(z)\) of DFS-rooted graphs \((G, T)\) with \(G\) in \(G\). We prove that \(H_G(z)\) is dominated by a series of the form \(D_{f_4(k)}(A(z))C_G(z)\), where \(f_4(k)\) is a certain function and \(D_{f_4(k)}(z)\) is as in Lemma 3.3. \(A(z)\) has radius of convergence 1/2, and \(C_G(z)\) is a series depending on \(G\). Then we show that the radius of convergence of \(C_G(z)\) is the inverse of \(\pi(\omega, \infty)\) if \(p(G) = \infty\), and the inverse of the maximum of \(\omega, \lambda(\ell(G)), \mu(m(G))\) and \(\pi(k, p_k(G))\) for \(k = 0, \ldots, \omega(G)\) otherwise.

Definition. Let \((G, T)\) be a DFS-rooted graph, and let \(r\) be the root of \(T\). An internal apex-path of \((G, T)\) is a path \(P = v_0, v_1, \ldots, v_{k+1}\) of \(T\), such that: (A) there exists a vertex \(w \notin P\) adjacent to both \(v_1\) and \(v_k\); and (B) for all \(i = 1, \ldots, k\) the vertex \(v_i\) is distinct from \(r\), and is adjacent to no vertex except \(v_{i-1}, v_{i+1}\) and possibly \(w\). Moreover, if the edge \((v_i, w)\) exists, it is not in \(T\).

The apex-reduction \((\tilde{G}, \tilde{T})\) of \((G, T)\) is obtained from \((G, T)\) by replacing every maximal internal apex-path \(v_0, v_1, \ldots, v_{k+1}\) by a vertex \(x\) adjacent to \(v_0, v_{k+1}\) and \(w\) (the edges \((v_0, x)\) and \((x, v_{k+1})\) are in \(\tilde{T}\) while \((x, w)\) is not); see Fig. 6.
Recall that $S$ is the set of rooted graphs defined in Eq. (3). An $S$-path is a path $P = v_0, v_1, \ldots, v_{k+1}$ of $T$ such that: (A) deleting the edges $(v_0, v_1)$ and $(v_k, v_{k+1})$ disconnects the path $P' = v_1, \ldots, v_k$ from $r$; and (B) the connected component containing $P'$ (which does not contain the root $r$) is an $S$-caterpillar $X(P)$ with spine $P'$.

The $S$-reduction $(\hat{G}, \hat{T})$ is obtained by replacing every maximal $S$-path $P = v_0, v_1, \ldots, v_{k+1}$ and the whole $S$-caterpillar $X(P)$ by the edge $(v_0, v_{k+1})$ (which is an edge in $\hat{T}$); see Fig. 6.

An example of apex- and $S$-reduction is shown in Fig. 7. Observe that an $S$-reduction can produce some multiple edges. More precisely, the reduced graph can have double edges (but no triple edges), with one edge in $\hat{T}$ and the other one not in $\hat{T}$.

**Lemma 4.7.** There exists a function $f_3(k)$ such that if $G$ does not contain any of $B_k, B'_k, D_k$ as a minor, and $T$ is a DFS-tree of $G$, then the apex-reduction $(\hat{G}, \hat{T})$ is such that for every $\hat{T}$-path $P$, the $P$-degree of every vertex is less than $f_3(k)$.

**Proof.** We prove the result for $f_3(k) = 12k(k+2f_0(k)) + 6$. Suppose that $w$ is a vertex of the apex-reduction $(\hat{G}, \hat{T})$ adjacent to $n = 12k(k+2f_0(k)) + 6$ vertices of $P$. The path $P$ may contain the root $r$, or/and the vertex $w$ or one vertex adjacent to $w$ by an edge of $\hat{T}$. However, there exists a subpath $P'$ of $P$ containing $n/3$ vertices adjacent to $w$, but neither $r$ nor $w$, nor any vertex adjacent to $w$ by an edge of $\hat{T}$. Let us denote $v_1, v_2, \ldots, v_{2m+2}$ the vertices adjacent to $w$ in order of appearance along $P'$, where $m = n/6 - 1 = 2k(k+2f_0(k))$.

Let $Q = x_0, \ldots, x_{j+1}$ be any path of $\hat{T}$ (with $j > 1$) such that $x_1$ and $x_j$ are both adjacent to $w$. By definition of apex-reduction, there must be a vertex $x$ in $\{x_1, \ldots, x_j\}$ which is either the root $r$, or adjacent to $w$ by an edge of $\hat{T}$, or adjacent to a vertex $w' \neq w$ by an edge not in $Q$; otherwise $Q$ would have been collapsed by the apex-reduction. In particular, for $i = 1, \ldots, m$, the subpath $P_i$ of $P'$ between $v_{2i}$ and $v_{2i+1}$ must contain a vertex $x_i$ adjacent to a vertex $w_i \neq w$ by an edge not in $P'$ ($x_i$ can be equal to $v_{2i}$ or $v_{2i+1}$). Observe that the subpaths $P_1, P_2, \ldots, P_m$ are all disjoint, so that the vertices $x_1, x_2, \ldots, x_m$ are all distinct. We now consider the multiset of vertices $W = \{w_1, \ldots, w_m\}$.

Suppose first that $2k$ vertices $w_1, \ldots, w_{2k}$ in $W$ coincide. The vertex $w' = w_1 = \cdots = w_{2k}$ may belong to $P'$, but we can suppose by symmetry that it does not appear before $x_{i_k}$ in $P'$. Now the minor $B'_k$ appears in the subgraph of $\hat{G}$ consisting of the edges joining $x_1, \ldots, x_k$ to $w'$, the edges joining $v_{2i_1-1}, \ldots, v_{2i_k-1}$ to $w$, and the subpath of $P'$ between $v_{i_1}$ and $x_{i_k}$. We reach a contradiction.
It follows that we can find a set $Y \subseteq W$ of $m/2k = (k + 2f_0(k))$ distinct vertices. If $k$ of them do not belong to $P'$, then the minor $B_k$ appears in the subgraph consisting of the path $P'$ together with the edges $(w, v_i)$ and the edges $(x_i, w_i)$; again a contradiction. Thus, $s = 2f_0(k)$ distinct vertices $y_1, \ldots, y_s$ in $V \subseteq W$ belong to $P'$. Let us denote $x_i_1, \ldots, x_i_s$ the vertices adjacent to $y_1, \ldots, y_s$, respectively. The edges $e_j = (x_{ij}, y_j)$ are $P'$-bridges for $j = 1 \ldots s$. Moreover the vertices $x_{i1}, \ldots, x_{is}$ are all distinct and so are $y_1, \ldots, y_s$. It can happen that $x_j = y_j'$ for $j \neq j'$, but there are at least $s/2 = f_0(k)$ $P'$-bridges with endpoints all distinct among $\{e_1, \ldots, e_s\}$.

This is impossible by Lemma 3.6.

**Proposition 4.8.** There exists a function $f_4(k)$ such that if $G$ does not contain any of the graphs $B_{k}, B_{k}'$, $D_{k}$, $Z_{1}^{k}$, $Z_{2}^{k}$, $Z_{3}^{k}$ as a minor and $T$ is a DFS-spanning tree, then the DFS-rooted graph $(\hat{G}, \hat{T})$ obtained by applying successively to $(G, T)$ the apex-reduction and then the $S$-reduction has height at most $f_4(k)$.

**Proof.** We prove the property for $f_4(k) = (2 + 8k)f_0(k)f_3(k) + 9k$. Let $P$ be a path of $\hat{T}$ in the reduction $(\hat{G}, \hat{T})$.

**Claim.** There are less than $2f_0(k)f_3(k)$ vertices (of $P$) incident to a $P'$-bridge.

**Proof.** Suppose the contrary. Then, Lemma 4.7 implies that there are $f_0(k)$ $P'$-bridges with endpoints all distinct. This is impossible by Lemma 3.7.

We say that a vertex $v$ not in $P$ is $2$-connected to $P$ if there are two disjoint paths from $v$ to $P$ (these paths share no other vertex than $v$).

**Claim.** There are less than $8kf_0(k)f_3(k)$ vertices in $P$ adjacent to a vertex $2$-connected to $P$.

**Proof.** Suppose there are $n = 8kf_0(k)f_3(k)$ vertices $v_1, \ldots, v_n$ in $P$ adjacent, respectively, to some vertices $v_1, \ldots, v_n$ (not in $P$) which are $2$-connected to $P$. By Lemma 4.7, the $P$-degree of any vertex is less than $f_3(k)$, hence there exists a subset $I$ of $\{1, \ldots, n\}$ of size $|I| = n/f_3(k) = 8kf_0(k)$ such that the vertices $v_i$ with $i \in I$ are all distinct. For every $i \in I$, the vertex $v_i$ is $2$-connected to $P$, hence there exists a simple path $P_i$ going from $v_i$ to a vertex $w_i \neq u_i$ in $P$. We can assume that $P_i$ contains no vertex of $P$ except $w_i$, and that if it contains $v_j$ for $j \neq i$, then $w_i = u_j$. The situation is represented in Fig. 8(a). We can find a subset $J \subset I$ of size $|J| = |I|/2 = 4kf_0(k)$ such that $w_i$ is distinct from $u_j$ for $i \neq j$ in $J$. Therefore, for all $i \neq j \in J$, the path $P_i$ contains neither $u_j$ nor $v_j$.

Let $K$ be a maximal subset of $J$ such that the paths $P_k$, $k \in K$ are all vertex-disjoint. By Lemma 4.7, $K$ has size less than $f_0(k)$ (indeed, contracting the paths in $K$ gives a set of $|K|$ $P'$-bridges with endpoints all distinct). By maximality of $K$, for any $j$ in $J \setminus K$ the path $P_j$ has a vertex in common with a path $P_i$ with $i \in K$. Hence, there exist $i$ in $K$ and $j' \subset (J \setminus K)$ of size $|j'| = (|J| - f_0(k))/f_0(k) = 4k - 1$ such that for all $j$ in $J'$ the paths $P_i$ and $P_j$ have a vertex in common. There is also a subset $j'' \subset J'$ of size $2k$ such that the vertices $u_j$, $j \in j''$ are all on the same side of $w_i$ on the path $P$. For all $j$ in $J''$, let $P'_{j}$ be a subpath of $P_j$ from $u_j$ to a vertex of $P_i$. Let $S$ be the subgraph of $\hat{G}$ made of the paths $P$, $P_i$ and $P'_{j}$, $j \in J''$, and the edges $(u_j, v_j)$, $j \in J''$; see Fig. 8(b). We now consider the minor of $\hat{G}$ obtained from $S$ by contracting all the edges of $P_i$ and all the edges of $P'_{j}$, $j \in J''$, except the edge incident to $v_j$. In this minor $P$ is a simple path (it remains unchanged) and all the vertices $v_j$, $j \in J''$ are distinct and adjacent to $w_i$ (recall that for every pair of distinct indices $j$, $j'$...
in \( J, v_j \) is not in \( P_j \). This minor clearly contains \( B'_{2k} \) (see Fig. 4), hence it contains \( B_k \). We reach a contradiction. 

A vertex of \( P \) is called a \( Z \)-vertex if it is neither incident to a \( P \)-bridge nor adjacent to a vertex (not in \( P \)) which is 2 connected to \( P \). Given the previous discussion, it suffices to prove that there are less than \( 9k \) \( Z \)-vertices.

Let \( v \) be a \( Z \)-vertex which is not one of the endpoints of \( P \). Since \( v \) has not been deleted during the \( S \)-reduction, \( v \) is incident to some other edges beside those in \( P \). Since \( v \) is neither incident to a \( P \)-bridge nor adjacent to a vertex 2-connected to \( P \), it is a cut vertex. More precisely, there is a 1-component \( C_v \) of \( G \) incident to \( v \), not containing any vertex of \( P \) and such that the graph \( C_v \) rooted at \( v \) does not belong to \( S \) (otherwise \( v \) would have been deleted during the \( S \)-reduction). By Lemma 4.1, the rooted graph \( C_v \) contains one of the graphs \( Z_i \) as a rooted minor. In order to conclude it suffices to observe that for \( i = 1, 2, 3 \) there are less than \( k \) vertices \( v \) such that \( C_v \) contains \( Z_i \) (otherwise, one of \( Z_k^1, Z_2^k \) or \( Z_3^k \) is a minor of \( \tilde{G} \)); and for \( i = 4, 5, 6 \) there are less than \( 2k \) vertices \( v \) such that \( C_v \) contains \( Z_i \) (otherwise, one of \( Z_1^k \) or \( Z_2^k \) is a minor of \( \tilde{G} \)). 

**Lemma 4.9.** Let \( k \) be an integer and let \( \mathcal{G} \) be a minor-closed class containing neither \( Z_1^k \) nor \( Z_2^k \) as minors. Then, the growth constant of \( \text{Cat}(S) \cap \mathcal{G} \) is at most \( \pi(\infty, \infty) \) if \( p(\mathcal{G}) = \infty \), and is at most the maximum of \( \lambda(\ell(\mathcal{G})), \mu(m(\mathcal{G})) \) and \( \pi(k, p(\mathcal{G})) \) for \( k = 0, \ldots, p(\mathcal{G}) \) otherwise.

**Proof.** We start with some notations. We define \( \mathbb{K} \equiv \mathbb{K}(\mathcal{G}) \) as the set of classes of rooted graphs consisting of the class \( \text{Cat}(S) \cap \mathcal{G} \), the class \( \mathcal{M}(m(\mathcal{G})) \), and either all the classes \( \mathcal{P}_{k,p(\mathcal{G})} \) for \( k = 0, \ldots, p(\mathcal{G}) \) if \( p(\mathcal{G}) = \infty \), or the class \( \mathcal{P}_{\infty,\infty} \) otherwise. We define \( \mathcal{X} \) as the set of rooted graphs defined as follows:

- If \( \ell(\mathcal{G}) \neq \infty \), then the rooted graph \( L_{\ell(\mathcal{G})+1} \) is in \( \mathcal{X} \).
- If \( m(\mathcal{G}) \neq \infty \), then the rooted graph \( M_{\ell(\mathcal{G})+1} \) is in \( \mathcal{X} \).
- If \( p(\mathcal{G}) \neq \infty \), then for all \( k = 0, \ldots, p(\mathcal{G}) \) the rooted graph \( P_{k,p(\mathcal{G})+1} \) is in \( \mathcal{X} \).

It is clear from the definitions that every rooted graph \( S \in S \) either contains a rooted minor in \( \mathcal{X} \) or belongs to one of the classes in \( \mathbb{K} \).

**Claim.** Let \( S, S' \) be rooted graphs in \( S \) not containing any minor in \( \mathcal{X} \). If the graph \( S \cdot S' \) obtained by gluing \( S \) and \( S' \) at their root-vertex does not belong to a class of \( \mathbb{K} \), then there is a graph \( X \) in \( \{Z_1, Z_2\} \cup \mathcal{X} \) which is a rooted minor \( S \cdot S' \).

**Proof.** If \( S \cdot S' \) does not belong to \( S \), then \( S \cdot S' \) contains \( Z_1 \) or \( Z_2 \) as a rooted minor. If \( S \cdot S' \) does belong to \( S \) but not to any class in \( \mathbb{K} \), then it contains a graph \( X \) in \( \mathcal{X} \) as a rooted minor.

**Claim.** There exists a constant \( N \) depending only on \( \mathcal{G} \) such that by removing at most \( N \) edges from any \( S \)-caterpillar in \( \mathcal{G} \) one can obtain a graph whose connected components are either graphs in \( S \) or \( K \)-caterpillars for a class \( K \) in \( \mathbb{K} \).

**Proof.** By definition, for any rooted graph \( X \) in \( \mathcal{X} \) there exists an integer \( d(X) \) such that \( X^{\cdot d(X)} \) is not in \( \mathcal{G} \). We prove the above property for \( N = 2(2k + \sum_{X \in \mathcal{X}} d(X)) \).

- Let \( G \) be an \( S \)-caterpillar in \( \mathcal{G} \). By definition \( G \) is made of a spine \( P = v_1, \ldots, v_p \) and rooted graphs \( S_1, \ldots, S_p \) in \( S \) attached to \( v_1, \ldots, v_p \) respectively. We define the integers \( 0 = i_0 < i_1 < \cdots < i_j = p \) as follows. If \( v_j \) is not a spine vertex, then \( i_j = i_{j+1} < p \) such that the graph obtained by gluing at their root-vertices all the rooted graphs \( S_i \) for \( i = i_j + 1, i_j + 2, \ldots, i_{j+1} \) contains a graph in \( \{Z_1, Z_2\} \cup \mathcal{X} \) as a rooted minor; if no such integer exists, then \( i_{j+1} = p \).
- Consider the graph \( G' \) obtained from \( G \) by deleting the edge \( (v_{j-1}, v_j) \) and the edges \( (v_j, v_{j+1}) \) for \( j = 1, \ldots, r-1 \). The connected components of \( G' \) are the graphs \( S_j \) for \( j = 1, \ldots, r \) (they belong to \( S \)) and the sub-caterpillars \( C_j \) containing the vertices \( v_i \) for \( i = i_j + 1, i_j + \ldots \).
2, \ldots, i_{j+1} - 1$, for the indices $j$ in $(1, \ldots, r)$ such that $i_j + 1 < i_{j+1}$. Let $j \in \{1, \ldots, r\}$ be such that $i_j + 1 < i_{j+1}$. By definition of $i_{j+1}$, gluing the rooted graphs $S_i$ for $i = i_j + 1, i_j + 2, \ldots, i_{j+1} - 1$ at their root-vertices gives a graph that contains none of the graphs in $\{Z_1, Z_2\} \cup \mathcal{X}$ as a minor. Therefore, the preceding claim implies that there is a class $\mathcal{K}$ in $\mathcal{K}$ such that the graphs $S_i$ for $i = i_j + 1, i_j + 2, \ldots, i_{j+1} - 1$ all belong to $\mathcal{K}$.

• It only remains to prove that $r$ is less than $2k + \sum_{X \in \mathcal{X}} d(X)$. Let $G''$ be the graph obtained from $G$ by contracting the subpaths of $P$ between $v_{i_j+1}$ and $v_{i_j}$ into a vertex $w_j$, for all $j = 1, \ldots, r$. The graph $G''$ is made of a path $w_1, \ldots, w_r$ incident to rooted graphs $R_1, \ldots, R_r$, respectively. Since each of the $R_i$ contains one of the graphs in $\{Z_1, Z_2\} \cup \mathcal{X}$ as a minor but $G''$ contains neither $Z_1^k, Z_2^k$ nor $X^d(X)$ for $X \in \mathcal{X}$, one must have $r < 2k + \sum_{X \in \mathcal{X}} d(X)$. \hfill \Box

Let $S(z)$ be the exponential generating function of $S$ and, for $\mathcal{K}$ in $\mathcal{K}$, let $C_{\mathcal{K}}(z)$ be the exponential generating function of $\mathcal{K}$-caterpillars. The generating function of the class $\mathcal{C}^*(z)$ of graphs whose connected components are either in $S$ or are $\mathcal{K}$-caterpillars for some $\mathcal{K}$ in $\mathcal{K}$ is equal to $\mathcal{C}^*(z) = \exp(S(z) - 1 + \sum_{\mathcal{K} \in \mathcal{K}} C_{\mathcal{K}}(z) - 1)$. Thus the growth constant of the class $\mathcal{C}^*$, which is the inverse of the radius of convergence of $\mathcal{C}^*(z)$, is at most $\pi(\infty, \infty)$ if $p(G') = \infty$, and is at most the maximum of $\lambda(\ell(G'))$, $\mu(m(G'))$ and $\pi(k, p_k(G'))$ for $k = 0, \ldots, p(G')$ otherwise. Moreover, by the argument above every graph in $\mathbb{Cat}(S) \cap G$ can be obtained by adding at most $N$ edges to a graph in $\mathcal{C}^*$. The number of ways of adding $N$ edges to a graph of size $n$ is polynomial in $n$ so the growth constants of $\mathbb{Cat}(S) \cap G$ and of $\mathcal{C}^*$ are the same. This concludes the proof. \hfill \Box

Proof of Theorem 4.4. Suppose $G$ is in $\text{Ex}(B_k, B'_k, D_k, Z_1^k, Z_2^k, Z_3^k)$. By Proposition 4.8, one obtains a DFS-rooted graph $(\widehat{G}, \widehat{T})$ with $\widehat{T}$ of height at most $f_4(k)$, by applying successively the apex- and $S$-reduction to any DFS-rooted graph $(G, T)$. An important remark is the following: for any DFS-rooted graph $(G, T)$ the $S$-paths are the same in $(G, T)$ and in the apex-reduction $(\widehat{G}, \widehat{T})$. Indeed, if $C$ is an $S$-caterpillar and $T$ is a spanning tree, then no vertex of $C$ is incident to exactly two edges in $T$ and one edge not in $T$. Therefore, for any minor closed class $G \subseteq \text{Ex}(B_k, B'_k, D_k, Z_1^k, Z_2^k, Z_3^k)$, one can obtain every graph in $G$ by choosing a DFS-rooted graph $(\widehat{G}, \widehat{T})$ with $\widehat{T}$ of height at most $f_4(k)$ and performing simultaneously the following two operations:

• Replace each edge of $\widehat{T}$ by an $S$-caterpillar in $\text{Cat}(S) \cap G$.

• Replace each vertex $x$ of $\widehat{G}$ adjacent to three vertices $u, v, w$, such that $(x, u)$ and $(x, v)$ are in $\widehat{T}$ and $(x, w)$ is not in $\widehat{T}$, by an internal apex-path $v_0 = u, v_1, v_2, \ldots, v_k, v = v_{k+1}$ where each vertex $v_i$ is joined to no vertex except $v_{i-1}, v_{i+1}$ and possibly $w$.

In terms of generating functions, the surjection described above shows that the GF $G(z)$ of the class $G$ is dominated by $D_{f_4(k)}(A(z)C_G(z))$ where $A(z) = 2z/(1 - 2z)$ is the GF of apex-paths and $C_G(z)$ is the GF of $\text{Cat}(S) \cap G$. By Lemma 4.9 the radius of convergence of $C_G(z)$ is at least the inverse of $\pi(\infty, \infty)$ if $p(G) = \infty$, and is at least the inverse of the maximum of $\lambda(\ell(G'))$, $\mu(m(G'))$ and $\pi(k, p_k(G'))$ for $k = 0, \ldots, p(G')$ otherwise. By the preceding remarks, this concludes the proof of Theorem 4.4. \hfill \Box

5. Concluding remarks and open problems

All our results concern labelled graphs. In the unlabeled setting, an analogue of the result from [22] holds: if $G$ is a minor-closed class of graphs and $u_n$ is the number of unlabeled members of $G \cap \mathbb{H}_n$, then $u_n \leq d^n$ for some constant $d$. This can be proved (we are grateful to Colin McDiarmid for this remark) using the fact [9] that graphs in a minor-closed class have a book-embedding with a bounded number of pages.

Given a proper minor-closed class of unlabeled graphs $\mathcal{U}$, its growth constant is defined as $\limsup(u_n)^{1/n}$. Using the techniques developed in this paper we can prove that the only unlabeled growth constants in the interval $[0, 2]$ are $0, 1$ and $2$. It must be noted that the growth constant for unlabeled caterpillars is equal to $2$. The gaps from Section 4 also give an infinity of gaps in the unlabeled case.
We conclude with some open problems.

1) We know that the growth constant of a class $G$ is the limit of $(g_n/n!)^{1/n}$ provided that all its excluded minors are 2-connected. The condition that the excluded minors are 2-connected is certainly not necessary as is seen by noting that the apex family of any class for which the limit exists also has a limit growth constant. It is also easy to see that such an apex family is also minor-closed and that at least one of its excluded minors is disconnected. The main open problem in this area seems to be whether $\lim (g_n/n!)^{1/n}$ exists for every minor-closed class $G$.

2) We have shown that $2^k$ is a growth constant for each non-negative integer $k$. A natural question is whether any other integer is a growth constant. More generally, is there any algebraic number in $\Gamma$ besides the powers of 2?

3) The growth constant of the class having $k+1$ (non-trivial) disjoint stars as forbidden minors is at least $2^k$, since it contains the $k$-th apex iteration of paths. Is it exactly $2^k$, as we have shown is the case for $k = 0, 1$?

Acknowledgments

We are very grateful to Pascal Ochem for his help in deriving the results in Section 4, to Colin McDiarmid for several useful remarks, including the suggestion of using the apex-construction and the significance of Ref. [9], to Angelika Steger for useful discussions, and to Norbert Sauer and Paul Seymour for information on well quasi-orders. We also thank the referee for useful suggestions.

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